Moduli problems of sheaves associated with oriented trees

Alexander Schmitt

February 1, 2008

Abstract

To every oriented tree we associate vector bundle problems. We define semistability concepts for these vector bundle problems and establish the existence of moduli spaces. As an important application, we obtain an algebraic construction of the moduli space of holomorphic triples.

Introduction

Let $Q=(V,A,t,h)$ be an oriented graph, or quiver, and X a fixed projective algebraic manifold. Associate to each vertex $i\in V$ a coherent sheaf $\mathscr{E}_i$ over X and to each arrow a homomorphism $\varphi_a\in \mathrm{Hom}(\mathscr{E}_{t(a)},\mathscr{E}_{h(a)})$, and let all the sheaves $\mathscr{E}_i$, $i\in V$, and homomorphisms φ_a, $a\in A$, vary. This yields interesting moduli problems in Algebraic Geometry.

To the author's knowledge, this kind of problems has been studied only in a few special cases: First, the results of King [8] cover the case when X is a point, second, the theory of semistable torsion free coherent sheaves deals with the quiver $\bullet$ (cf. [7]). Apart from this, the objects associated with $\bullet \longrightarrow \bullet$ when X is a curve, the so-called holomorphic triples (which we will review below) were treated by Bradlow and Garcia-Prada [3], [5], the objects associated with A_n-quivers ($\bullet \longrightarrow \cdots \longrightarrow \bullet$) have been studied under the name of holomorphic chains in the context of dimensional reduction by Álvarez-Cónsul and Gracía-Prada [1], and, finally, Higgs bundles or Hitchin pairs [6], [15], [21], [23], [18] can be seen as the objects associated with the quiver consisting of one vertex joined by an arrow to itself. More recently, Álvarez-Cónsul and Gracía-Prada have announced generalizations of their work [1] to quivers without oriented cycles.

In this note, we address the case when X is an arbitrary projective manifold and Q is an oriented tree. One of the central points of the paper is the definition of ϑ-semistability for representations of Q, i.e., tuples $(\mathscr{E}_i, i\in V;\varphi_a, a\in A)$ as above. The notion we propose is obtained by applying principles from our paper [20]. It also fits in the framework suggested by King [8] for finding the semistability concept. The main result of our paper is the existence of moduli spaces for the ϑ-semistable representations. The construction of the moduli spaces itself applies the general GIT machinery of [11], adapted to vector bundle problems by Seshadri, Gieseker, Maruyama, Simpson and others (see [7] and [20] for precise references). Of course, due to the complexity of the objects we study, the details become technically quite involved.

The motivation to study these questions is the following: First, the case of $X=\{\mathrm{pt}\}$, i.e., King's work, is important because of its relation to the classification theory of algebras

[4], [17]. Recently, King's moduli spaces have found applications to non-commutative algebraic geometry [9], and generalizations of them were applied to the theory of quantum algebras [10], [13], [14]. One might therefore hope that the moduli spaces we construct will find applications in one or the other of these areas.

Second, there is the special case of $\bullet \longrightarrow \bullet$ or, more generally, the A_n-quivers. Suppose X is a smooth, projective curve. A holomorphic triple is then a triple (E_1, E_2, φ) consisting of two vector bundles E_1 and E_2 on X and a homomorphism φ between them. These objects were obtained by Bradlow and Garcia-Prada in [3] and [5] by a process of dimensional reduction from certain $\mathrm{SU}(2)$-equivariant bundles on $X \times \mathbb{P}_1$. They defined the notion of τ-(semi)stability for holomorphic triples and gave a construction of the moduli space of τ-stable triples. In [3], it was suggested to construct this moduli space via Geometric Invariant Theory. Of course, the notion of a holomorphic triple has an obvious interpretation on higher dimensional manifolds, and our results contain the construction on an arbitrary projective manifold X. The central technical point in our GIT construction is the identification of the semistable points in a suitable parameter space which is trickier than usual due to the failure of a certain addivity property of the weights for the actions one has to study. To solve this, we will prove a decomposition theorem for one parameter subgroups of $\mathrm{SL}(V) \times \mathrm{SL}(W)$ w.r.t. the action on $\mathrm{Hom}(V, W)$. It might also be interesting to note that the GIT construction reveals that there are actually two parameters involved in the definition of semistability for holomorphic triples. This might become useful for relating the moduli space of holomorphic triples to other moduli spaces.

Finally, the results of this paper are after the results of the paper [20] a further step towards a universal theory working for (almost) all vector bundle problems. In such a theory, the input would be a representation of a (reductive) algebraic group G on a finite dimensional vector space W — which defines in a natural way a moduli problem —, and the output would be the (parameter dependent) semistability concept and moduli spaces for the semistable objects. For a precise formulation in the case $G = \mathrm{GL}(r)$ and a solution of the moduli problem over curves, we refer the reader to our paper [20]. The present paper corresponds to the case of $G = \prod_{i \in V} \mathrm{GL}(r_i)$ with its representation on $\bigoplus_{a \in A} \mathrm{Hom}(\mathbb{C}^{r_{t(a)}}, \mathbb{C}^{r_{h(a)}})$.

We work only with oriented trees, because they are well suited for inductions. Indeed, they can be thought of as being built "inductively", just like a tree in the real world grows from a small tree to a big tree, spreading more and more branches. Therefore, many of the problems reduce to the quiver $\bullet \longrightarrow \bullet$. Notably, an inductive procedure can be used to extend the already mentioned decomposition result which is the key to the general theory. At the moment, our techniques do not extend to arbitrary quivers, and examples suggest that indeed further technical difficulties will arise for quivers other than oriented trees.

Notations and conventions

The base field is the field of complex numbers. (The restriction to characteristic zero is necessary to apply Maruyama's boundedness result ([7], Thm. 3.3.7) for certain families of torsion free coherent sheaves.)

In the sequel, X is understood to be a projective manifold, and $\mathscr{O}_X(1)$ to be an ample invertible sheaf on X. For any coherent sheaf $\mathscr{E}$, we write $P(\mathscr{E})$ for its Hilbert polynomial w.r.t. $\mathscr{O}_X(1)$, and degrees and slopes are computed w.r.t. $\mathscr{O}_X(1)$. Most of the time, we will fix some polynomials P_i, i in some index set. Having done this, r_i, d_i, and μ_i are used for the rank, degree, and slope defined by these polynomials when interpreted as Hilbert polynomials w.r.t. $\mathscr{O}_X(1)$. Recall that any torsion free coherent sheaf $\mathscr{G}$ has a uniquely

determined slope Harder-Narasimhan filtration $0 = \mathscr{G}_0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}_l = \mathscr{G}$, such that all the quotients $\mathscr{G}_i/\mathscr{G}_{i-1}$, $i = 1,\dots,l$, are slope semistable, and $\mu(\mathscr{G}_i/\mathscr{G}_{i-1}) > \mu(\mathscr{G}_{i+1}/\mathscr{G}_i)$, $i = 1,\dots,l-1$ (see [7], §1.6). One sets $\mu_{\max}(\mathscr{G}) := \mu(\mathscr{G}_1)$ and $\mu_{\min}(\mathscr{G}) := \mu(\mathscr{G}/\mathscr{G}_{l-1})$ Finally, $\chi(\mathscr{G}) = P(\mathscr{G})(0)$ stands for the Euler characteristic of $\mathscr{G}$.

We use Grothendieck's convention for projectivizing a vector bundle E, i.e., $\mathbb{P}(E)$ is the bundle of hyperplanes in the fibres of the vector bundle E.

Acknowledgments

The paper was written during the author's stays at the University of Barcelona and at the Bar-Ilan University. During this time, the author was supported by grant #1996SGR00060 of the Generalitat de Catalunya and by a grant of the Emmy-Noether Insitute, respectively.

1 Statement of the main results

Representations of quivers

We first recall the notion of a representation of a quiver in an abelian category and describe an abstract semistability concept, going back to King [8].

A *quiver* Q consists of a set $V := \{1,\dots,n\}$ whose elements are called *vertices* and a set A, the *arrows*, together with maps $h,t: A \longrightarrow V$. Given an arrow $a \in A$, we call $h(a)$ its *head* and $t(a)$ its *tail*.

Convention. We will only deal with quivers without multiple arrows. Therefore, we often write an arrow in the form $(t(a),h(a))$.

Example 1.1. Given a quiver Q as above, the *underlying graph* is the graph Γ_Q whose set of vertices is just V and whose set of edges is $E := \{\{i_1,i_2\} \mid (i_1,i_2) \in A\}$. An *oriented tree* is a quiver Q with connected underlying graph and $\#V = \#A + 1$. This means precisely that the underlying graph Γ_Q is a tree.

A *path in Q from i to j* is a sequence of arrows $a_1,\dots,a_m$ with $t(a_1) = i$, $h(a_l) = t(a_{l+1})$, $l = 1,\dots,m-1$, and $h(a_m) = j$. Here, m is called *the length of the path*. Moreover, for every vertex i, one adds a path $(i|i)$ of length zero, joining i to itself. A quiver Q now defines an additive category $\underline{\mathrm{Q}}$ where all objects are direct sums of indecomposables and the indecomposable objects are the elements of V, and, for $i, j \in V$, $\mathrm{Mor}_{\underline{\mathrm{Q}}}(i,j) =$ Set of paths from i to j.

Given an abelian category $\underline{\mathrm{A}}$, a *representation of Q in $\underline{\mathrm{A}}$* is a covariant additive functor from $\underline{\mathrm{Q}}$ to $\underline{\mathrm{A}}$. Two representations $\underline{\mathrm{R}}$ and $\underline{\mathrm{R}}'$ are called *equivalent*, if they are isomorphic as functors. We denote by $\underline{\mathrm{Rep}}_{\underline{\mathrm{A}}}(Q)$ the abelian category of all representations of Q in $\underline{\mathrm{A}}$. Notions like *sub-representations*, *quotient representations*, *direct sums of representations*, etc., are then defined in the usual way.

Remark 1.2. A representation $\underline{\mathrm{R}}$ of Q in $\underline{\mathrm{A}}$ is specified by a collection $\mathscr{E}_i$, $i \in V$, of objects in $\underline{\mathrm{A}}$ and a collection of morphisms $\varphi_a \in \mathrm{Mor}_{\underline{\mathrm{A}}}(\mathscr{E}_{t(a)},\mathscr{E}_{h(a)})$, $a \in A$. We then write simply $\underline{\mathrm{R}} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$. Note that two representations $\underline{\mathrm{R}} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ and $\underline{\mathrm{R}}' = (\mathscr{E}'_i, i \in V; \varphi'_a, a \in A)$ are equivalent, if and only if there are isomorphisms $\psi_i: \mathscr{E}_i \longrightarrow \mathscr{E}'_i$, $i \in V$, with $\varphi'_a = \psi_{h(a)} \circ \varphi_a \circ \psi_{t(a)}^{-1}$ for all arrows $a \in A$.

Semistability

Let $(G,\leq)$ be a totally ordered abelian group and $\vartheta\colon \mathrm{Ob}(\underline{\mathrm{Rep}}_{\underline{\mathrm{A}}}(Q)) \longrightarrow G$ a map which factorizes over a group homomorphism $K_{\underline{\mathrm{A}}}(Q) \longrightarrow G$. Here, $K_{\underline{\mathrm{A}}}(Q)$ is the K-group of the abelian category $\underline{\mathrm{Rep}}_{\underline{\mathrm{A}}}(Q)$.

Then, a representation $\underline{\mathrm{R}}$ is called *ϑ-(semi)stable*, if and only if the following two conditions are satisfied

1. $\vartheta(\underline{\mathrm{R}}) = 0$
2. $\vartheta(\underline{\mathrm{R}}') \quad (\leq) \quad 0$ for every non-trivial proper sub-representation $\underline{\mathrm{R}}'$ of $\underline{\mathrm{R}}$.

Recall that "$(\leq)$" means that "$<$" is used for defining "stable" and "$\leq$" for defining "semistable".

Representations of quivers in Algebraic Geometry

Let Q be a quiver and $(X,\mathscr{O}_X(1))$ a polarized projective manifold defined over $\mathbb{C}$. In this paper, we will consider representations of Q in the category $\underline{\mathrm{Coh}}(X)$ of coherent sheaves on X.

Convention. In the following, the word "representation" refers to a representation in $\underline{\mathrm{Coh}}(X)$.

Let $\underline{\mathrm{R}} := (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ be a representation of Q. The function

$$\begin{array}{rcl} \underline{\mathrm{P}}\colon V & \longrightarrow & \mathbb{Q}[x] \\ i & \longmapsto & P(\mathscr{E}_i), \end{array}$$

is called the *type of* $\underline{\mathrm{R}}$.

Let $\underline{\mathrm{P}}\colon V \longrightarrow \mathbb{Q}[x]$ be a map and S a noetherian scheme. A *family of representations of type* $\underline{\mathrm{P}}$ *of Q parametrized by S* is a tuple $(\mathfrak{E}_{S,i}, i \in V; \varphi_{S,a}, a \in A)$ consisting of S-flat families $\mathfrak{E}_{S,i}$ of coherent sheaves with Hilbert polynomials $\underline{\mathrm{P}}(i)$ on $S \times X$, $i \in V$, and elements $\varphi_{S,a} \in \mathrm{Hom}\big(\mathfrak{E}_{S,t(a)}, \mathfrak{E}_{S,h(a)}\big)$, $a \in A$. We leave it to the reader to define *equivalence of families*.

Semistability

Convention. From now on, we assume that Q is an oriented tree.

We have to find a "good" function ϑ for which we can prove explicit results. For this, the totally order abelian group will be $(\mathbb{Q}[x],\leq)$ where "$\leq$" is the lexicographic order of polynomials. We will now give the definition of ϑ and explain at the end of this section why it is the natural choice. The definition of ϑ depends on several parameters, namely,

- a function $\underline{\mathrm{P}}\colon V \longrightarrow \mathbb{Q}[x] \setminus \{0\}$,
- a collection $\underline{\sigma}_Q = (\sigma_i, i \in V)$ of positive rational polynomials σ_i, $i \in V$, of degree at most $\dim X - 1$, and
- a collection $\underline{b}_Q = (b_a, a \in A)$ of positive rational numbers b_a, $a \in A$.

Having fixed these data, we write $P_i := \underline{\mathrm{P}}(i)$, and let r_i be the associated rank, $i \in V$. We also set $\sigma := \sigma_1 \cdot \ldots \cdot \sigma_n$ and $\check{\sigma}_i := \sigma/\sigma_i$, $i \in V$.

Now define $\vartheta := \vartheta(\underline{\mathrm{P}}, \underline{\sigma}_Q, \underline{b}_Q)$ as the function which assigns to a representation $\underline{\mathrm{R}} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ the polynomial

$$\sum_{a\in A} b_a \quad \Bigg[\check{\sigma}_{t(a)} \Big\{ P(\mathscr{E}_{t(a)}) - \mathrm{rk}\,\mathscr{E}_{t(a)} \frac{P_{t(a)} - \sigma_{t(a)}}{r_{t(a)}} \Big\} \\ + \check{\sigma}_{h(a)} \Big\{ P(\mathscr{E}_{h(a)}) - \mathrm{rk}\,\mathscr{E}_{h(a)} \frac{P_{h(a)} + \sigma_{h(a)}}{r_{h(a)}} \Big\} \Bigg].$$

Properties of ϑ-semistable representations

We the above definition of ϑ, we have the concept of ϑ-(semi)stability for representations of Q at hand. Here is a list of properties of this concept. In the following $\underline{\mathrm{R}} := (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ is assumed to be a ϑ-semistable representation, unless otherwise mentioned.

1. The condition $\vartheta(\underline{\mathrm{R}}) = 0$ is automatic if the type of $\underline{\mathrm{R}}$ is $\underline{\mathrm{P}}$.

2. The sheaves $\mathscr{E}_i$ must all be torsion free. Indeed, set $\mathscr{F}_i := \mathrm{Tors}(\mathscr{E}_i)$, $i \in V$, and $\varphi'_a := \varphi_{a|\mathscr{F}_{t(a)}}$, $a \in A$. Then, $\underline{\mathrm{R}}' := (\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ is a sub-representation of $\underline{\mathrm{R}}$ with

$$\vartheta(\underline{\mathrm{R}}') \quad = \quad \sum_{a\in A} b_a \big[\check{\sigma}_{t(a)} P(\mathscr{F}_{t(a)}) + \check{\sigma}_{h(a)} P(\mathscr{F}_{h(a)}) \big].$$

   This polynomial is strictly positive as soon as one of the sheaves $\mathscr{F}_i$ is non-trivial.

3. Suppose the type of $\underline{\mathrm{R}}$ is $\underline{\mathrm{P}}$. Then all homomorphisms φ_a, $a \in A$, are non-zero. If, say, φ_{a_0} were zero, we could remove the arrow a_0 from Q in order to obtain two disjoint subtrees $Q_{t(a_0)}$ and $Q_{h(a_0)}$. We define $\mathscr{F}_i$ as $\mathscr{E}_i$ if $i \in V_{t(a_0)}$ and as 0 otherwise, and $\varphi'_a := \varphi_{a|\mathscr{F}_{t(a)}}$ for all $a \in A$. Thus, $\underline{\mathrm{R}}' = (\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ is a sub-representation of $\underline{\mathrm{R}}$ with

$$\vartheta(\underline{\mathrm{R}}') \quad = \quad b_{a_0} \cdot \sigma \cdot r_{t(a_0)} \quad > \quad 0,$$

   contradicting the assumption that $\underline{\mathrm{R}}$ be ϑ-semistable.

4. The representation $\underline{\mathrm{R}}$ possesses a *Jordan-Hölder filtration*

$$0 =: \underline{\mathrm{R}}^{(m+1)} \subset \underline{\mathrm{R}}^{(m)} \subset \cdots \subset \underline{\mathrm{R}}^{(1)} \subset \underline{\mathrm{R}}^{(0)} := \underline{\mathrm{R}}$$

   where $\underline{\mathrm{R}}^{(l)}$ is a sub-representation of $\underline{\mathrm{R}}^{(l-1)}$ which is maximal w.r.t. inclusion among those sub-representations $\underline{\mathrm{R}}'$ with $\vartheta(\underline{\mathrm{R}}') = 0$, $l = 1, \ldots, m+1$. The successive quotients $\underline{\mathrm{R}}^{(l)}/\underline{\mathrm{R}}^{(l+1)}$, $l = 0, \ldots, m$, are thus ϑ-stable representations and the *associated graded object*

$$\mathrm{gr}(\underline{\mathrm{R}}) \quad := \quad \bigoplus_{l=0}^{m} \underline{\mathrm{R}}^{(l)}/\underline{\mathrm{R}}^{(l+1)}$$

   which is well-defined up to equivalence is again a ϑ-semistable representation of the same type as $\underline{\mathrm{R}}$.

   As usual, two ϑ-semistable representations $\underline{\mathrm{R}}$ and $\underline{\mathrm{R}}'$ are called *S-equivalent*, if their associated graded objects are equivalent, and $\underline{\mathrm{R}}$ is called *ϑ-polystable*, if it is equivalent to $\mathrm{gr}(\underline{\mathrm{R}})$.

5. One can "join" ϑ-semistable representations: Let Q_1 and Q_2 be subquivers of Q, such that $Q = Q_1 \cup Q_2$ and $A_1 \cap A_2 = \varnothing$, and $\underline{\mathrm{R}} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ a — not necessarily ϑ-semistable — representation of Q. Suppose the representation $\underline{\mathrm{R}}_j := (\mathscr{E}_i, i \in V_j, \varphi_a, a \in A_j)$ is $\vartheta(\underline{\mathrm{P}}_j, \underline{\sigma}_{Q_j}, \underline{b}_{Q_j})$-(semi)stable for $j = 1,2$. Then, $\underline{\mathrm{R}}$ is ϑ-(semi)stable. Here, the data $(\underline{\mathrm{P}}_j, \underline{\sigma}_{Q_j}, \underline{b}_{Q_j})$ are obtained from $(\underline{\mathrm{P}}, \underline{\sigma}_Q, \underline{b}_Q)$ by restriction to Q_j, $j = 1,2$.

Example 1.3 (Holomorphic triples). A *holomorphic triple* is a triple $(\mathscr{E}_1, \mathscr{E}_2, \varphi)$ consisting of two coherent sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ and a homomorphism $\varphi: \mathscr{E}_1 \longrightarrow \mathscr{E}_2$. In other words, a holomorphic triple is a representation of the quiver $\bullet \longrightarrow \bullet$. Specializing our general definitions to holomorphic triples, we say that — for given positive polynomials σ_1 and $\sigma_2 \in \mathbb{Q}[x]$ of degree at most $\dim X - 1$ — a holomorphic triple $(\mathscr{E}_1, \mathscr{E}_2\, \varphi)$ is ϑ-(semi)stable, if for any two subsheaves $\mathscr{F}_1$ and $\mathscr{F}_2$ of $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively, such that $0 \neq \mathscr{F}_1 \oplus \mathscr{F}_2 \neq \mathscr{E}_1 \oplus \mathscr{E}_2$ and $\varphi(\mathscr{F}_1) \subset \mathscr{F}_2$

$$\sigma_2 \left(P(\mathscr{F}_1) - \mathrm{rk}\, \mathscr{F}_1 \left(\frac{P(\mathscr{E}_1)}{\mathrm{rk}\, \mathscr{E}_1} - \frac{\sigma_1}{\mathrm{rk}\, \mathscr{E}_1} \right) \right) + \sigma_1 \left(P(\mathscr{F}_2) - \mathrm{rk}\, \mathscr{F}_2 \left(\frac{P(\mathscr{E}_2)}{\mathrm{rk}\, \mathscr{E}_2} + \frac{\sigma_2}{\mathrm{rk}\, \mathscr{E}_2} \right) \right)$$

is a (non-positive) negative polynomial in the lexicographic order of polynomials. Here, ϑ is associated with $i \longmapsto P(\mathscr{E}_i)$, σ_1, σ_2, and $b_a = 1$.

If X is a curve and $\sigma_1 = \sigma_2 =: \sigma \in \mathbb{Q}_+$, set $\tau := \mu(\mathscr{E}_2) + \sigma / \mathrm{rk}\, \mathscr{E}_2$. Then the above definition yields the definition of τ-(semi)stability of Bradlow and Garcia-Prada [3], [5] for holomorphic triples. Thus, we see that our concept of semistability is more general because it involves two parameters instead of one. This might be useful for comparing the moduli spaces of holomorphic triples with other moduli spaces.

Remark 1.4. Let $\underline{\mathrm{R}} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ be a representation of type $\underline{\mathrm{P}}$. By property 5., the ϑ-(semi)stability condition is satisfied, if, for every arrow $a \in A$, $(\mathscr{E}_{t(a)}, \mathscr{E}_{h(a)}, \varphi_a)$ is a ϑ_a-(semi)stable holomorphic triple, ϑ_a being obtained by restricting the data $(\underline{\mathrm{P}}, \underline{\sigma}_Q, \underline{b}_Q)$ to the subquiver $t(a) \longrightarrow h(a)$, $a \in A$. Therefore, if all σ_i's are equal to some σ, existence theorems for ϑ-(semi)stable representations on curves can be extracted from the work of Bradlow and Garcia-Prada [3].

Example 1.5. Assume that our base manifold X is just a point. Then, $\underline{\mathrm{Coh}}(X)$ is the category of finite dimensional complex vector spaces, i.e., a representation is of the form $(E_i, i \in V; f_a, a \in A)$ where E_i is a finite dimensional $\mathbb{C}$-vector space, $i \in V$, and $f_a: E_{t(a)} \longrightarrow E_{h(a)}$ is a linear map, $a \in A$.

In this case, the datum $\underline{\sigma}_Q$ is obsolete (strictly speaking, not defined). This means, we just fix $\underline{\mathrm{P}}: V \longrightarrow \mathbb{Z}_{>0}$ and $\underline{b}_Q = (b_a, a \in A)$ and define

$$\vartheta(E_i, i \in V; f_a, a \in A) \quad := \quad \sum_{a \in A} b_a \left(\frac{\dim E_{t(a)}}{\underline{\mathrm{P}}(t(a))} - \frac{\dim E_{h(a)}}{\underline{\mathrm{P}}(h(a))} \right).$$

The corresponding concept of ϑ-(semi)stability agrees with King's notion of χ-(semi)stability for representations $\underline{\mathrm{R}}$ with dimension vector (=type) $\underline{\mathrm{P}}$ associated to the character

$$\begin{array}{rccc} \chi: & \mathrm{GL}(\underline{\mathrm{P}}(1)) \times \cdots \times \mathrm{GL}(\underline{\mathrm{P}}(n)) & \longrightarrow & \mathbb{C}^* \\ & (m_1, \ldots, m_n) & \longmapsto & \det(m_1)^{s_1} \cdot \ldots \cdot \det(m_n)^{s_n}, \end{array}$$

with

$$s_i \quad := \quad \sum_{a:t(a)=i} \frac{b_a}{\underline{\mathrm{P}}(i)} - \sum_{a:h(a)=i} \frac{b_a}{\underline{\mathrm{P}}(i)}, \qquad i = 1, \ldots, n.$$

This shows that our definition is a natural extension of King's (specialized to oriented trees) to higher dimensions.

The main result

Fix $\underline{\mathrm{P}}$, $\underline{\sigma}_Q$, and $\underline{b}_Q$ as before, and set $\vartheta := \vartheta(\underline{\mathrm{P}}, \underline{\sigma}_Q, \underline{b}_Q)$. Define $\underline{\mathrm{M}}(Q)^{\vartheta-(s)s}_{\underline{\mathrm{P}}}$ as the functor which assigns to a noetherian scheme S the set of equivalence classes of families of ϑ-(semi)stable representations of type $\underline{\mathrm{P}}$ of Q which are parametrized by S.

Theorem 1.6. i) *There exist a projective scheme $\mathscr{M} := \mathscr{M}(Q)^{\vartheta-ss}_{\underline{\mathrm{P}}}$ and a natural transformation $\underline{\mathrm{T}}: \underline{\mathrm{M}}(Q)^{\vartheta-ss}_{\underline{\mathrm{P}}} \longrightarrow h_{\mathscr{M}}$, such that for any other scheme $\mathscr{M}'$ and any natural transformation $\underline{\mathrm{T}}'$, there exists a unique morphism $\rho: \mathscr{M} \longrightarrow \mathscr{M}'$ with $\underline{\mathrm{T}}' = h(\rho) \circ \underline{\mathrm{T}}$.*

ii) *The map $\underline{\mathrm{T}}(\mathrm{pt})$ induces a bijection between the set of S-equivalence classes of ϑ-semistable representations of Q of type $\underline{\mathrm{P}}$ and the set of closed points of $\mathscr{M}$.*

iii) *The space $\mathscr{M}$ contains an open subscheme $\mathscr{M}(Q)^{\vartheta-s}_{\underline{\mathrm{P}}}$ which becomes through $\underline{\mathrm{T}}$ a coarse moduli scheme for the functor $\underline{\mathrm{M}}(Q)^{\vartheta-s}_{\underline{\mathrm{P}}}$.*

Example 1.7. As an illustration how such a moduli space is constructed and as a tool for later sections, we review the case $X = \{\mathrm{pt}\}$, i.e., King's construction in the case Q is an oriented tree.

Fix $\underline{\mathrm{P}}: V \longrightarrow \mathbb{Z}_{>0}$, $\underline{b}_Q$, and write $\vartheta := \vartheta(\underline{\mathrm{P}}, \underline{b}_Q)$, and let χ be the corresponding character (see 1.5) of

$$\mathrm{GL}(\underline{\mathrm{P}}) \quad := \quad \mathrm{GL}(\underline{\mathrm{P}}(1)) \times \cdots \times \mathrm{GL}(\underline{\mathrm{P}}(n)).$$

Every representation of Q with dimension vector $\underline{\mathrm{P}}$ is equivalent to one in the space

$$\mathfrak{H}(\underline{\mathrm{P}}) \quad := \quad \bigoplus_{a \in A} \mathrm{Hom}\big(\mathbb{C}^{\underline{\mathrm{P}}(t(a))}, \mathbb{C}^{\underline{\mathrm{P}}(h(a))}\big).$$

On this space, there is a natural left action of $\mathrm{GL}(\underline{\mathrm{P}})$, and the set of equivalence classes of representations with dimension vector $\underline{\mathrm{P}}$ corresponds to the set of $\mathrm{GL}(\underline{\mathrm{P}})$-orbits in $\mathfrak{H}(\underline{\mathrm{P}})$. However, this set does not carry a natural structure of an algebraic variety or even of a topological space.

In order to obtain an algebraic variety, we must use the GIT machinery, i.e., we must choose a linearization of the given action in $\mathscr{O}_{\mathfrak{H}(\underline{\mathrm{P}})}$. This is given by the character χ. King's moduli space is the GIT quotient $\mathfrak{H}(\underline{\mathrm{P}}) /\!\!/_{\chi} \mathrm{GL}(\underline{\mathrm{P}})$. Note that one must show that the points which are (semi)stable w.r.t. the given linearization are exactly the ones corresponding to χ-(semi)stable representations.

We can describe the quotient in another way. Note that we know (Property 3.) that every homomorphism f_a occuring in a ϑ-semistable representation $(E_i, i \in V; f_a, a \in A)$ must be non-zero. Moreover, one has

Lemma 1.8. *Let Q be an oriented tree and $(f_a, a \in A)$ be a point in $\mathfrak{H}(\underline{\mathrm{P}})$. Then, for a given $a_0 \in A$ and $z \in \mathbb{C}^*$, there exists an element $g_{a_0,z} = (z_1\,\mathrm{id}, \ldots, z_n\,\mathrm{id}) \in \mathrm{GL}(\underline{\mathrm{P}})$ such that $(f'_a, a \in A) := g_{a_0,z} \cdot (f_a, a \in A)$ looks as follows: $f'_a = f_a$ for $a \neq a_0$, and $f'_{a_0} = z \cdot f_{a_0}$.*

Proof. Removing the arrow a_0 from the quiver Q yields two connected subquivers $Q_{t(a_0)}$ and $Q_{h(a_0)}$. We set $z_i := z$ for $i \in V_{h(a_0)}$ and $= 1$ for $i \in V_{t(a_0)}$. Then, $g_{a_0,z} := (z_1 \,\mathrm{id}, \dots, z_n \,\mathrm{id})$ does the trick.

Thus, we can start with the space

$$\mathfrak{P}(\underline{\mathrm{P}}) \quad := \quad \prod_{a \in A} \mathbb{P}\left(\mathrm{Hom}\big(\mathbb{C}^{\underline{\mathrm{P}}(t(a))}, \mathbb{C}^{\underline{\mathrm{P}}(h(a))}\big)^{\vee}\right)$$

with fixed ample line bundle $\mathscr{O}(b_a, a \in A)$. The moduli space is now the projective variety $\mathfrak{P}(\underline{\mathrm{P}})/\!\!/_{\underline{b}_Q}\mathrm{SL}(\underline{\mathrm{P}})$. Here, it will follow from Theorem 3.3 that the $\mathscr{O}(b_a, a \in A)$-(semi)stable points are exactly those which correspond to ϑ-(semi)stable representations. It is easy to see that this is the same variety as $\mathfrak{H}(\underline{\mathrm{P}})/\!\!/_{\chi}\mathrm{GL}(\underline{\mathrm{P}})$.

Now, let X be an arbitrary projective manifold. The GIT construction in this case follows the same pattern:

- Step 1: Find a variety $\mathfrak{T}$ analogous to $\mathfrak{P}(\underline{\mathrm{P}})$ which parametrizes representations of type $\underline{\mathrm{P}}$ and contains every ϑ-semistable representation at least once.
- Step 2: Show that there is an action of a reductive algebraic group G, such that two points in $\mathfrak{T}$ lie in the same orbit if and only if they correspond to equivalent representations.
- Step 3: Find a linearization, such that a point in $\mathfrak{T}$ is (semi)stable w.r.t. that linearization if and only if it corresponds to a ϑ-(semi)stable representation.

Having treated all these steps successfully, one gets the moduli space as the GIT quotient $\mathfrak{T}/\!\!/G$. In our construction, the assumption that Q be an oriented tree is essential. First, it allows us as in the example to choose G as a product of special linear groups. This makes the computations for the Hilbert-Mumford criterion already simpler. Second, for the action of $\mathrm{SL}(\underline{\mathrm{P}})$ on $\mathfrak{P}(\underline{\mathrm{P}})$, one has Theorem 3.3 which simplifies the computations even further. But this theorem is also crucial for proving that one can in fact adjust all parameters appearing in such a way that Step 1 - 3 really go through.

Concluding Remarks

How did we find ϑ?

The most general moduli problem one would like to treat is the following: Let G be a reductive algebraic group and Y a projective manifold on which G acts. If P is a principal G-bundle on X, we obtain an induced fibre space $Y(P) := P \times^{G} Y$. One would now like to classify pairs (P, σ) where P is a principal G-bundle and $\sigma: X \longrightarrow Y(P)$ is a section. For this, one has to define a general semistability concept and establish the existence of moduli spaces. In gauge theory, such a programme has been succesfully treated in [2] and [12]. In the algebraic context, the author has defined this semistability concept and constructed the moduli spaces in the case when X is a curve, $G = \mathrm{GL}(r)$, and the action of the center $\mathbb{C}^* \cdot E_n \subset G$ on Y is trivial. The semistability concept is a version of the Hilbert-Mumford criterion and depends only on the choice of a linearization of the G-action on Y [20]. This concept is completely natural and reproduces all known examples.

In the present paper, we have $G = \mathrm{GL}(\underline{\mathrm{P}})$ and $Y = \mathfrak{P}(\underline{\mathrm{P}})$. Note that the induced $\mathbb{C}^{*n}$-action on Y is trivial. One can adapt the techniques of [20] to the present situation. We illustrate this by an example:

Let us look at the case when X is a curve and $Q = 1 \longrightarrow 2 \longrightarrow 3$. The natural parameter space for a bounded family of triples of vector bundles (E_1, E_2, E_3) where $\deg E_i =: d_i$ and $\mathrm{rk} E_i =: r_i$ are fixed, $i = 1, 2, 3$, is a product of quot-schemes $\mathfrak{Q} := \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3$. If we polarize $\mathfrak{Q}_i$ by the line bundle $\mathscr{O}_i(1)$ coming from Gieseker's covariant map, we can polarize $\mathfrak{Q}$ by $\mathscr{O}(\tau_1, \tau_2, \tau_3)$ where the τ_i are positive rational numbers. We choose in addition a polarization $\mathscr{O}(b_1, b_2)$ on $\mathbb{P}(\mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})^\vee) \times \mathbb{P}(\mathrm{Hom}(\mathbb{C}^{r_2}, \mathbb{C}^{r_3})^\vee)$.

Let $\underline{\mathrm{R}} := (E_1, E_2, E_3; \varphi_1, \varphi_2)$ be a representation of Q. As explained in [20], the objects for testing semistability are triples

$$\underline{T} = \big((\underline{E}^{1,\bullet}, \underline{\alpha}_1), (\underline{E}^{2,\bullet}, \underline{\alpha}_2), (\underline{E}^{3,\bullet}, \underline{\alpha}_3) \big),$$

where

$$(\underline{E}^{i,\bullet}, \underline{\alpha}_i) = \big(0 \subset E^{i,1} \subset \cdots \subset E^{i,s_i} \subset E, (\alpha_{i,1}, \ldots, \alpha_{i,s_i}) \big)$$

is a weighted filtration for E_i, $i = 1, 2, 3$. Recall that

$$M\big(\underline{E}^{i,\bullet}, \underline{\alpha}_i\big) \quad := \quad \sum_{j=1}^{s_i} \alpha_{i,j} \big(\deg E_i \, \mathrm{rk} E^{i,j} - \deg E^{i,j} r_i \big), \quad i = 1, 2, 3.$$

One defines $\mu_{b_1, b_2}(\underline{T}; \varphi_1, \varphi_2)$ similarly as in [20].

With these conventions, $\underline{\mathrm{R}}$ is called $(\tau_1, \tau_2, \tau_3; b_1, b_2)$-*(semi)stable*, if for every triple of weighted filtrations $\underline{T}$ as above one finds

$$\sum_{i=1}^{3} \tau_i M\big(\underline{E}^{i,\bullet}, \underline{\alpha}_i\big) + \mu_{b_1, b_2}\big(\underline{T}; \varphi_1, \varphi_2\big) \quad (\geq) \quad 0.$$

Now, the decomposition results of Section 3 permit us to restrict to triples $\underline{T}$ where

$$(\underline{E}^{i,\bullet}, \underline{\alpha}_i) \quad = \quad \big(0 \subset F_i \subset E, (1/r_i) \big)$$

for some subbundles F_i of E_i, $i = 1, 2, 3$, with $\varphi_1(F_1) \subset F_2$ and $\varphi_2(F_2) \subset F_3$. In this case, one has

$$\mu_{b_1, b_2}\big(\underline{T}; \varphi_1, \varphi_2\big) \quad = \quad b_1 \left(\frac{\mathrm{rk} F_2}{r_2} - \frac{\mathrm{rk} F_1}{r_1} \right) + b_2 \left(\frac{\mathrm{rk} F_3}{r_3} - \frac{\mathrm{rk} F_2}{r_2} \right),$$

so that the condition becomes

$$\begin{aligned} \tau_1 \big(\mathrm{rk} F_1 \mu(E_1) - \deg F_1 \big) + \tau_2 \big(\mathrm{rk} F_2 \mu(E_2) - \deg F_2 \big) + \tau_3 \big(\mathrm{rk} F_3 \mu(E_3) - \deg F_3 \big) \\ + b_1 \left(\frac{\mathrm{rk} F_2}{r_2} - \frac{\mathrm{rk} F_1}{r_1} \right) + b_2 \left(\frac{\mathrm{rk} F_3}{r_3} - \frac{\mathrm{rk} F_2}{r_2} \right) \quad (\geq) \quad 0. \end{aligned}$$

Now, write $\tau_1 = b_1/\sigma_1$, $\tau_2 = (b_1 + b_2)/\sigma_2$, and $\tau_3 = b_2/\sigma_3$ for some positive rational numbers σ_i. We thus see that $\underline{\mathrm{R}}$ will be $(\tau_1, \tau_2, \tau_3; b_1, b_2)$-(semi)stable, if and only if for every sub-representation $(F_1, F_2, F_3; \varphi_1', \varphi_2')$, such that $F_1 \oplus F_2 \oplus F_3$ is a non-trivial proper subbundle of $E_1 \oplus E_2 \oplus E_3$, one has

$$\begin{aligned} & b_1 \left(\frac{1}{\sigma_1} \Big(\mathrm{rk} F_1 \frac{d_1 - \sigma_1}{r_1} - \deg F_1 \Big) + \frac{1}{\sigma_2} \Big(\mathrm{rk} F_2 \frac{d_1 + \sigma_2}{r_2} - \deg F_2 \Big) \right) \quad + \\ & b_2 \left(\frac{1}{\sigma_2} \Big(\mathrm{rk} F_2 \frac{d_2 - \sigma_2}{r_2} - \deg F_2 \Big) + \frac{1}{\sigma_3} \Big(\mathrm{rk} F_3 \frac{d_3 + \sigma_3}{r_3} - \deg F_3 \Big) \right) \quad (\geq) \quad 0. \end{aligned}$$

Multiply this by $-\sigma_1 \sigma_2 \sigma_3$ to recover our original definition.

Other quivers

One would expect to be able to treat other quivers as well, at least when they don't contain oriented cycles. In this case, the concept of semistability should depend on $\underline{\mathrm{P}}$ and $\underline{\sigma}_Q$ as before and a character χ of the group $\mathrm{GL}(\underline{\mathrm{P}}(1)) \times \cdots \times \mathrm{GL}(\underline{\mathrm{P}}(n))$, but I expect it to look much more difficult.

For quivers with oriented cycles, there arise other problems. Look for example at the theory of Higgs bundles which can be viewed as the moduli problem associated with the quiver consisting of one vertex and an arrow, joining the vertex to itself. Applying the above definition of a representation would lead to the consideration of sheaves $\mathscr{E}$ together with an endomorphism $\varphi: \mathscr{E} \longrightarrow \mathscr{E}$. It turns out that the (semi)stability concept forces $\mathscr{E}$ to be a (semi)stable sheaf (see [18], Thm. 3.1, with $L = \mathscr{O}_X$). As, moreover, a stable sheaf has no endomorphisms besides multiples of the identity, this theory brings nothing new. The way out is to consider twisted endomorphisms $\varphi: \mathscr{E} \longrightarrow \mathscr{E} \otimes L$, L a suitable line bundle, e.g., $L = K_X$, if X is a curve of genus $g \geq 2$ [6].

Finally, the resulting moduli spaces are only quasi-projective. In order to compactify them, one must add further data [18], [20]. If one does this, one finds also semistability concepts which cannot be formulated as conditions on sub-representations only [20].

2 More notation concerning quivers

We introduce now some terminology for quivers which is adapted to the subsequent proofs.

Let $Q = (V, A, t, h)$ be a quiver. A *subquiver* $Q' \subset Q$ consists of a subset $V' \subset V$ and a subset $A' \subset A$, such that $h(A') \cup t(A') \subset V'$, and is called a *full subquiver*, if any arrow $a \in A$ with $h(a) \in V'$ and $t(a) \in V'$ lies in A'. Obviously, we can associate to any subset $V' \subset V$ a full subquiver $Q(V')$ of Q. For any $i \in V$, *the star of i* is defined as the subquiver $\mathrm{Star}_Q(i)$ of Q whose set of arrows is $A(i) := \{\, a \in A \,|\, h(a) = i \,\vee\, t(a) = i \,\}$ and whose set of vertices is $V(i) := h(A(i)) \cup t(A(i))$. A vertex i of the subquiver Q' is called an *end (of Q' in Q)*, if $\mathrm{Star}_Q(i)$ is not contained in Q'. The set of all ends will be denoted by $\mathrm{END}_Q(Q')$. For each vertex $i \in \mathrm{END}_Q(Q')$, the set of *ingoing arrows* $\mathrm{In}_Q(i)$ is defined as the set of all arrows of $\mathrm{Star}_Q(i)$ not lying in Q' whose head is i. Similarly, we define $\mathrm{Out}_Q(i)$, the set of *outgoing arrows*.

The following lemma will enable us to prove many of the needed technical details by induction.

Lemma 2.1. *Let Q be an oriented tree, then there exists a vertex i whose star is either $\big(\{i,i'\},(i,i')\big)$ or $\big(\{i,i'\},(i',i)\big)$ for some vertex $i' \in V$.*

So, after relabelling the vertices, we can assume that $i = n$ and $i' = n-1$ and define a new quiver Q' with $V' := \{\,1,\ldots,n-1\,\}$ and $A' = A \setminus \{(n-1,n)\}$ or $A' = A \setminus \{(n,n-1)\}$. This is again an oriented tree.

3 Decomposition of one parameter subgroups

In this section, we prove the main auxiliary result which simplifies the Hilbert-Mumford criterion for the actions we consider. At a first reading, the reader might follow this Section till Theorem 3.3 and then proceed directly to the proof of the main result.

Let Q be an oriented tree with $V = \{1,...,n\}$. Let $V_1,...,V_n$, and $W_1,...,W_n$ be finite dimensional $\mathbb{C}$-vector spaces and suppose we are given representations $\tau_i\colon \mathrm{SL}(V_i) \longrightarrow \mathrm{GL}(W_i)$, $i = 1,...,n$. Set $\underline{\tau} := (\tau_1,...,\tau_n)$, define $\underline{\mathrm{P}}\colon V \longrightarrow \mathbb{Z}_{>0}$, $i \longmapsto p_i := \dim V_i$, and $\mathfrak{P}(\underline{\mathrm{P}}) := \prod_{a\in A} \mathbb{P}(\mathrm{Hom}(V_{t(a)}, V_{h(a)})^\vee)$. These data define an action of $\mathrm{SL}(\underline{\mathrm{P}}) := \prod_{i\in V} \mathrm{SL}(V_i)$ on

$$\mathbb{P}_{\underline{\tau},\underline{\mathrm{P}}} \quad := \quad \mathbb{P}(W_1^\vee) \times \cdots \times \mathbb{P}(W_n^\vee) \times \mathfrak{P}(\underline{\mathrm{P}}).$$

Fix a (fractional) polarization $\mathscr{O}(l_1,...,l_n; b_a, a \in A)$ on the space $\mathbb{P}_{\underline{\tau},\underline{\mathrm{P}}}$ where the l_i and b_a are positive rational numbers. It will be our task to describe the (semi)stable points in $\mathbb{P}_{\underline{\tau},\underline{\mathrm{P}}}$ w.r.t. the given linearization.

Further assumptions and notations

A one parameter subgroup of $\mathrm{SL}(\underline{\mathrm{P}})$ will be written as $\lambda = (\lambda_1,....,\lambda_n)$ where λ_i is a one parameter subgroup of $\mathrm{SL}(V_i)$, $i = 1,...,n$. Let $\underline{w} := ([w_1],...,[w_n]; [f_a], a \in A)$ be a point in $\mathbb{P}_{\underline{\tau},\underline{\mathrm{P}}}$ and λ be a one parameter subgroup of $\mathrm{SL}(\underline{\mathrm{P}})$. Then, $\mu(\underline{w},\lambda)$ is defined as minus the weight of the induced $\mathbb{C}^*$-action on the fibre of $\mathscr{O}(l_1,...,l_n, b_a, a \in A)$ over $\underline{w}_\infty := \lim_{z\to\infty} \lambda(z)\cdot\underline{w}$. Recall that the Hilbert-Mumford critrion states that $\underline{w}$ is (semi)stable w.r.t. the linearization in $\mathscr{O}(l_1,...,l_n; b_a, a\in A)$ if and only if $\mu(\underline{w},\lambda)\ (\geq)\ 0$ for all non trivial one parameter subgroups λ of $\mathrm{SL}(\underline{\mathrm{P}})$.

Now, let $\underline{w}$ and λ be as before. For $i = 1,...,n$, write $\mu([w_i],\lambda_i)$ for the weight of the $\mathbb{C}^*$-action induced by the action of λ_i on the fibre of $\mathscr{O}_{\mathbb{P}(W_i^\vee)}(-1)$ over the point $\lim_{z\to\infty} \lambda_i(z)[w_i]$, and, for an arrow $a \in A$, we let $\mu([f_a],(\lambda_{t(a)},\lambda_{h(a)}))$ be the weight of the resulting $\mathbb{C}^*$-action on the fibre of $\mathscr{O}_{\mathbb{P}(\mathrm{Hom}(V_{t(a)},V_{h(a)})^\vee)}(-1)$ over $\lim_{z\to\infty}(\lambda_{t(a)}(z),\lambda_{h(a)}(z))\cdot[f_a]$. With these conventions

$$\mu(\underline{w},\lambda) \quad = \quad l_1\mu([w_1],\lambda_1)+\cdots+l_n\mu([w_n],\lambda_n)+\sum_{a\in A} b_a\mu\big([f_a],(\lambda_{t(a)},\lambda_{h(a)})\big).$$

Next, we remind you that a one parameter subgroup λ_i of $\mathrm{SL}(V_i)$ is defined by giving a basis $v_1^i,...,v_{p_i}^i$ for V_i and integer weights $\gamma_1^i \leq \cdots \leq \gamma_{p_i}^i$ with $\sum_j \gamma_j^i = 0$. We will also use *formal one parameter subgroups* which are defined as before, only that this time the γ_j^i are allowed to be rational numbers. It is clear how to define $\mu(\underline{w},\lambda)$ for a formal one parameter subgroup. Let $\lambda = (\lambda_1,...,\lambda_n)$ be a (formal) one parameter subgroup where λ_i is given (w.r.t. to some basis of V_i) by the weight vector $\underline{\gamma}^i$. Then, we write $\underline{\gamma} = (\underline{\gamma}^1,...,\underline{\gamma}^n)$. For $j = 0,...,p_i$, we can look at the one parameter subgroup $\lambda_i^{(j)}$ which is defined by the weight vector $\gamma^{i,(j)} := (j-p_i,...,j-p_i,j,...,j)$, $j-p_i$ occuring j times. Note that both $\lambda_i^{(0)}$ and $\lambda_i^{(p_i)}$ are the trivial one parameter subgroup. The $\lambda_i^{(j)}$ with $1 \leq j < p_i$ are particularly important due to the fact that any weight vector $\underline{\gamma}^i = (\gamma_1^i,...,\gamma_{p_i}^i)$ as before can be written as

$$\underline{\gamma}^i \quad = \quad \sum_{j=1}^{p_i-1} \frac{(\gamma_{j+1}^i - \gamma_j^i)}{p_i}\gamma^{i,(j)}. \tag{1}$$

Assumptions 3.1. We require the following additivity property for the action of $\mathrm{SL}(V_i)$ on $\mathbb{P}(W_i^\vee)$, $i = 1,...,n$: For every point $[w_i] \in \mathbb{P}(W_i^\vee)$, every basis $v_1^i,...,v_p^i$ of W_i, and every two one parameter subgroups λ and λ' of $\mathrm{SL}(V_i)$ which are given with respect to that basis by weight vectors $(\gamma_1,...,\gamma_p)$ and $(\gamma_1',...,\gamma_p')$ with $\gamma_1 \leq \cdots \leq \gamma_p$ and $\gamma_1' \leq \cdots \leq \gamma_p'$, we have

$$\mu([w_i],\lambda\cdot\lambda') \quad = \quad \mu([w_i],\lambda)+\mu([w_i],\lambda').$$

Example 3.2. In general, one has in the above situation only "$\le$", e.g., for the action of $\mathrm{SL}(V)$ on $\mathrm{End}(V)$. To see this, consider for example $V=\mathbb{C}^3$ and the action of $\mathrm{SL}_3(\mathbb{C})$ on $M_3(\mathbb{C})$, the vector space of complex (3×3)-matrices, by conjugation. Let f be given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and λ and λ' with respect to the standard basis by $(-2,1,1)$ and $(-1,-1,2)$, respectively. Then, $\mu([f],\lambda)=0=\mu([f],\lambda')$, but $\mu([f],\lambda\cdot\lambda')=-3$. A similar phenomenon is responsible for the technicalities which we will encounter below.

If we are given bases $v^i_1,\dots,v^i_{p_i}$ for V_i, we set $V_i^{(j)}:=\langle v^i_1,\dots,v^i_j\rangle$, $j=0,\dots,p_i$, $i=1,\dots,n$. Let $\underline{j}:=(j_1,\dots,j_n)$ be a tuple of elements with $j_i\in\{0,\dots,p_i\}$, $i=1,\dots,n$. To such a tuple we associate the weight vector

$$\underline{\gamma}^{\underline{j}} \quad := \quad \Bigl(\frac{1}{p_1}\gamma^{1,(j_1)},\dots,\frac{1}{p_n}\gamma^{n,(j_n)}\Bigr),$$

and denote the corresponding formal one parameter subgroup of $\mathrm{SL}(\underline{\mathrm{P}})$ by $\lambda^{\underline{j}}$.

Theorem 3.3. *In the above setting, assume that there are positive rational numbers $\alpha_{i,a}$, $i\in V$, $a\in A$, with*

$$l_i \quad = \sum_{a\in A:t(a)=i\vee h(a)=i} b_a\alpha_{i,a},$$

then a point $([w_1],\dots,[w_n],[f_a],a\in A)$ is (semi)stable w.r.t. the linearization of the action of $\mathrm{SL}(\underline{\mathrm{P}})$ on $\mathbb{P}_{\tau,\underline{\mathrm{P}}}$ in $\mathscr{O}(l_1,\dots,l_n;b_a,a\in A)$, if and only if for all possible choices of bases $v^i_1,\dots,v^i_{p_i}$ for V_i, $i\in V$, and indices $j_i\in\{0,\dots,p_i\}$ with

$$f_a\bigl(V_{t(a)}^{(j_{t(a)})}\bigr) \quad\subset\quad V_{h(a)}^{(j_{h(a)})}, \qquad \textit{for all } a\in A,$$

one has

$$0 \quad (\le) \quad \sum_{a\in A} b_a\Bigl[\alpha_{t(a),a}\frac{1}{p_{t(a)}}\mu\bigl([w_{t(a)}],\lambda_{t(a)}^{(j_{t(a)})}\bigr)-\frac{j_{t(a)}}{p_{t(a)}} + \alpha_{h(a),a}\frac{1}{p_{h(a)}}\mu\bigl([w_{h(a)}],\lambda_{h(a)}^{(j_{h(a)})}\bigr)+\frac{j_{h(a)}}{p_{h(a)}}\Bigr].$$

The rest of this Section concerns the proof of Theorem 3.3. Let $([f_a],a\in A)$ be an element in $\mathfrak{P}(\underline{\mathrm{P}})$. We call the weight vector $\underline{\gamma}^{\underline{j}}$ *basic (w.r.t. $([f_a],a\in A)$)*, if (1) the subquiver $Q_{\underline{j}}:=Q(\{i\in V\,|\,0<j_i<p_i\})$ is connected, (2) for any arrow $a\in A_{\underline{j}}$ we have $f_a\bigl(V_{t(a)}^{(j_{t(a)})}\bigr)\subset V_{h(a)}^{(j_{h(a)})}$, and (3) neither $V_{t(a)}^{(j_{t(a)})}\subset\ker f_a$ nor $V_{h(a)}^{(j_{h(a)})}\supset\operatorname{Im} f_a$, $a\in A$. The strategy is now to decompose the weight vector of any given one parameter subgroup in a suitable way into basic ones, so that a point will be (semi)stable if and only if the Hilbert-Mumford criterion is satisfied for basic formal one parameter subgroups.

The central decomposition theorem

The case $\#V = 2$

Let $[f] \in \mathfrak{P}(\underline{P})$ be the class of a homomorphism $f: V_1 \longrightarrow V_2$. Given bases $v_1^j, \ldots, v_{p_j}^j$ for V_j, $j = 1, 2$, we write $f = \sum_{i,j} f_{i,j} v_i^{1\vee} \otimes v_j^2$.

Theorem 3.4. *Let $[f]$ be as before fixed. Then, for any given one parameter subgroup (λ_1, λ_2) of $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$ which is specified by the bases $v_1^1, \ldots, v_{p_j}^j$ of V_j, $j = 1, 2$, and the weight vector $(\underline{\gamma}^1, \underline{\gamma}^2)$, there exist indices $i(1)_*$ and $i(2)_*$ with $f_{i(1)_* i(2)_*} \neq 0$ such that $\mu([f], (\lambda_1, \lambda_2)) = \mu([v^1_{i(1)_*}{}^\vee \otimes v^2_{i(2)_*}], (\lambda_1, \lambda_2))$ and a decomposition*

$$(\underline{\gamma}^1, \underline{\gamma}^2) \quad = \sum_{\substack{i(1)=0,\ldots,p_1-1;\\ i(2)=0,\ldots,p_2-1}} \eta_{i(1)i(2)} \Big(\frac{1}{p_1} \gamma^{1,(i(1))}, \frac{1}{p_2} \gamma^{2,(i(2))} \Big),$$

such that all the coefficients are non-negative rational numbers, and whenever the coefficient $\eta_{i(1)i(2)}$ is not zero, the weight vector $(\gamma^{1,(i(1))}, \gamma^{2,(i(2))})$ is a basic weight vector and $\mu([f], \lambda^{(i(1),i(2))}) = \mu([v^1_{i(1)_*}{}^\vee \otimes v^2_{i(2)_*}], \lambda^{(i(1),i(2))})$.

One infers

$$\begin{aligned}
& \mu\big(([w_1],[w_2],[f]),(\lambda_1,\lambda_2)\big) \\
= \; & \mu\big(([w_1],[w_2],[v^1_{i(1)_*}{}^\vee \otimes v^2_{i(2)_*}]),(\lambda_1,\lambda_2)\big) \\
= \; & \sum \eta_{i(1)i(2)} \mu\big(([w_1],[w_2],[v^1_{i(1)_*}{}^\vee \otimes v^2_{i(2)_*}]),\lambda^{(i(1),i(2))}\big) \\
= \; & \sum \eta_{i(1)i(2)} \mu\big(([w_1],[w_2],[f]),\lambda^{(i(1),i(2))}\big),
\end{aligned}$$

whence we have achieved our goal in this situation.

Proof of Theorem 3.4. To reduce indices, we slightly change the notation: We set $V := V_1$ and $W := V_2$. The dimensions of V and W are denoted by p and q, respectively. One parameter subgroups of $\mathrm{SL}(V)$ will denoted by the letter κ, weight vectors used for defining a one parameter subgroup of $\mathrm{SL}(V)$ will be denoted by $\underline{\delta} = (\delta_1, \ldots, \delta_p)$. For one parameter subgroups of $\mathrm{SL}(W)$ we use λ, and write weight vectors as $\underline{\gamma} = (\gamma_1, \ldots, \gamma_q)$. If we are given bases $v_1, \ldots, v_p$ for V and $w_1, \ldots, w_q$ for W, $i \in \{0, \ldots, p\}$ and $j \in \{0, \ldots, q\}$, we set $V^{(i)} := \langle 1, \ldots, i \rangle$ and $W^{(j)} := \langle 1, \ldots, j \rangle$.

Now, let (κ, λ) be an arbitrary one parameter subgroup of $\mathrm{SL}(V) \times \mathrm{SL}(W)$. Choose bases $v_1, \ldots, v_p$ of V and $w_1, \ldots, w_q$ of W with respect to which κ and λ act diagonally and are determined by weight vectors $(\delta_1, \ldots, \delta_p)$ with $\delta_1 \le \cdots \le \delta_p$, $\sum \delta_i = 0$, and $(\gamma_1, \ldots, \gamma_q)$ with $\gamma_1 \le \cdots \le \gamma_q$, $\sum \gamma_j = 0$, respectively. Set $j_0 := \min\{ j \mid \mathrm{Im}\, f \subset W^{(j)} \}$, $i_0 := \min\{ i \mid f(V^{(i)}) \not\subset W^{(j_0-1)} \}$, $i_0' := \min\{ i \mid V^{(i)} \not\subset \ker f \}$, and $j_0' := \min\{ j \mid f(V^{(i_0')}) \subset W^{(j)} \}$. Write $f = \sum f_{i,j} v_i^\vee \otimes w_j$. Let s, t be indices such that $f_{s,t} \neq 0$. For $i = 1, \ldots, p$, $m(i; s, t)$ denotes the weight of the eigenvector $v_s^\vee \otimes w_t$ with respect to the action of the one parameter subgroup $\kappa^{(i)}$. In the same way, the numbers $n(j; s, t)$ are defined. Then, one easily checks:

Lemma 3.5. i) $\mu([f], (\kappa^{(i)}, \lambda^{(0)})) = m(i; s, t)$ *unless $i_0' \le i < s$. In that case, we will have $\mu([f], (\kappa^{(i)}, \lambda^{(0)})) = p - i$ and $m(i; s, t) = -i$.*

ii) $\mu([f], (\kappa^{(0)}, \lambda^{(j)})) = n(j; s, t)$ *unless $t \le j < j_0$. Then, $\mu([f], (\kappa^{(0)}, \lambda^{(j)})) = j$ and $n(j; s, t) = j - q$.*

Theorem 3.4 can now be restated as

Theorem. *In the above situation, there exist indices i_* and j_* with $f_{i_*,j_*} \neq 0$ and a decomposition of the weight vector*

$$
\begin{aligned}
(\underline{\delta},\underline{\gamma}) \quad = \quad & \sum_{i=1}^{i_0'-1} \alpha_i \delta^{(i)} \quad + \quad \sum_{i=i_*}^{p-1} \widetilde{\alpha}_i \delta^{(i)} \quad + \quad \sum_{j=j_0}^{q-1} \beta_j \gamma^{(j)} \quad + \\
& + \quad \sum_{j=1}^{j_*-1} \widetilde{\beta}_j \gamma^{(j)} \quad + \quad \sum_{\substack{i=i_0',\dots,p-1;\\ j=1,\dots,j_0-1}} \eta_{i,j} \Big(\frac{1}{p} \delta^{(i)}, \frac{1}{q} \gamma^{(j)} \Big).
\end{aligned}
$$

The α_i, $\widetilde{\alpha}_i$, β_j, $\widetilde{\beta}_j$, and $\eta_{i,j}$ are non-negative rational numbers such that $\eta_{i,j} = 0$ whenever (a) $f(V^{(i)}) \not\subset W^{(j)}$, *or* (b) $i < i_*$ *and* $j \geq j_*$, *or* (c) $i \geq i_*$ *and* $j < j_*$.

Lemma 3.6. *Let $\underline{\delta} = \sum_{i=i_1}^{i_2} \alpha_i \delta^{(i)}$, $\alpha_i \in \mathbb{Q}_{\geq 0}$, $i = i_1, \dots, i_2$, and $\underline{\gamma} = \sum_{j=j_1}^{j_2} \beta_j \gamma^{(j)}$, $\beta_j \in \mathbb{Q}_{\geq 0}$, $j = j_1, \dots, j_2$, be vectors of rational numbers.*

i) *If $\sum_{i=i_1}^{i_2} p\alpha_i \geq \sum_{j=j_1}^{j_2} q\beta_j$, then we can decompose the vector $(\underline{\delta},\underline{\gamma})$ in the following way*

$$
(\underline{\delta},\underline{\gamma}) \quad = \quad \sum_{i=i_1}^{i_2} \alpha_i' \delta^{(i)} + \sum_{\substack{i=i_1,\dots,i_2;\\ j=j_1,\dots,j_2}} \eta_{i,j} \Big(\frac{1}{p} \delta^{(i)}, \frac{1}{q} \gamma^{(j)} \Big)
$$

where all the α_i' and $\eta_{i,j}$ are non-negative rational numbers, $\alpha_i' \leq \alpha_i$, $i = i_1, \dots, i_2$, and $\eta_{i,j} = 0$ when $\alpha_i \cdot \beta_j = 0$. Moreover, if equality holds in the assumption, all the α_i' are zero.

ii) *For $\sum_{i=i_1}^{i_2} p\alpha_i \leq \sum_{j=j_1}^{j_2} q\beta_j$, we can write*

$$
(\underline{\delta},\underline{\gamma}) \quad = \quad \sum_{j=j_1}^{j_2} \beta_j' \gamma^{(j)} + \sum_{\substack{i=i_1,\dots,i_2;\\ j=j_1,\dots,j_2}} \eta_{i,j} \Big(\frac{1}{p} \delta^{(i)}, \frac{1}{q} \gamma^{(j)} \Big)
$$

where all the β_j' and $\eta_{i,j}$ are non-negative rational numbers, $\beta_j' \leq \beta_j$, $j = j_1, \dots, j_2$, and $\eta_{i,j} = 0$ when $\alpha_i \cdot \beta_j = 0$. Furthermore, if equality is assumed, the β_j' are zero.

Proof. Obvious.

Let $\delta_{\mathrm{i}_1} < \dots < \delta_{\mathrm{i}_e}$ be the different weights which appear, and $V := \bigoplus_{\varepsilon=1}^{e} V^{\delta_{\mathrm{i}_\varepsilon}}$ be the corresponding decomposition of V into eigenspaces. We will write $V^{\leq \delta_{\mathrm{i}_\varepsilon}} := \bigoplus_{\varepsilon'=1}^{\varepsilon} V^{\delta_{\mathrm{i}_{\varepsilon'}}}$ and $V^{< \delta_{\mathrm{i}_\varepsilon}} := \bigoplus_{\varepsilon'=1}^{\varepsilon-1} V^{\delta_{\mathrm{i}_{\varepsilon'}}}$. Set $\alpha_i := (1/p)(\delta_{i+1} - \delta_i)$, $i = 1, \dots, p-1$.

Similarly, let $\gamma_{\mathrm{j}_1} < \dots < \gamma_{\mathrm{j}_f}$ be the different weights occuring. This gives rise to analogous constructions as before which we will not write down explicitly. Define $\beta_j := (1/q)(\gamma_{j+1} - \gamma_j)$, $j = 1, \dots, q-1$.

We have to make the computation at an eigenvector $v_{i_*}^\vee \otimes w_{j_*}$ with $f_{i_*,j_*} \neq 0$ at which $\mu([f],(\kappa,\lambda))$ is achieved. So, we first look at all weights which appear as weights of eigenvectors appearing non-trivially in the decomposition of f. Hence, we define

$$
\begin{aligned}
G := \Big\{ (\varepsilon,\sigma) \in \{1,\dots,e\} \times \{1,\dots,f\} \quad | \quad & \exists\, \hat{\imath} \in \{1,\dots,p\},\ \hat{\jmath} \in \{1,\dots,q\} : \\
& f_{\hat{\imath},\hat{\jmath}} \neq 0 \ \wedge\ v_{\hat{\imath}} \in V^{\delta_{\mathrm{i}_\varepsilon}} \ \wedge\ w_{\hat{\jmath}} \in W^{\gamma_{\mathrm{j}_\sigma}} \Big\}.
\end{aligned}
$$

Let $\gamma_{\mathfrak{k}_1} < \cdots < \gamma_{\mathfrak{k}_{l_0}}$ be the weights with $\mathfrak{k}_\iota = \mathfrak{j}_\sigma$ for some $(\varepsilon,\sigma) \in G$, $\iota = 1,\dots,l_0$. Set $\mathfrak{h}_{l_0} := \min\{\,\mathfrak{i}_\varepsilon \,|\, (\varepsilon,\sigma) \in G$ with σ such that $\mathfrak{k}_{l_0} = \mathfrak{j}_\sigma\,\}$. If $f(V^{<\delta_{\mathfrak{h}_{l_0}}}) = \{0\}$, we stop. Otherwise, we define l_1 by

$$\mathfrak{k}_{l_1} = \max\{\,\mathfrak{j}_\sigma \,|\, (\varepsilon,\sigma) \in G \text{ with } \varepsilon \text{ such that } \mathfrak{i}_{\varepsilon+1} \le \mathfrak{h}_{l_0}\,\},$$

i.e., $\mathfrak{k}_{l_1}$ is determined by the requirements $\varphi\big(V^{<\delta_{\mathfrak{h}_{l_0}}}\big) \subset W^{\le \gamma_{\mathfrak{j}_\sigma}}$ and that there be an $(\varepsilon,\sigma) \in G$ with $\mathfrak{j}_\sigma = \mathfrak{k}_{l_1}$ and $\mathfrak{i}_\varepsilon < \mathfrak{h}_{l_0}$. Note that $l_1 < l_0$. Next, set $\mathfrak{h}_{l_1} := \min\{\,\mathfrak{i}_\varepsilon \,|\, (\varepsilon,\sigma) \in G$ with σ such that $\mathfrak{k}_{l_1} = \mathfrak{j}_\sigma\,\}$. Now, iterate this process to get — after relabelling — weights $\delta_{\mathfrak{h}_1} < \cdots < \delta_{\mathfrak{h}_l}$ and $\gamma_{\mathfrak{k}_1} < \cdots < \gamma_{\mathfrak{k}_l}$ with the property that

$$f\big(V^{<\delta_{\mathfrak{h}_1}}\big) \;=\; \{0\},\;\; f\big(V^{<\delta_{\mathfrak{h}_\iota}}\big) \;\subset\; W^{\le \gamma_{\mathfrak{k}_{\iota-1}}},\; \iota = 2,\dots,l, \;\text{ and } f(V) \;\subset\; W^{\le \gamma_{\mathfrak{k}_l}}. \tag{2}$$

Finally, we proceed to the proof. We choose $* \in \{\,1,\dots,l\,\}$ such that $-\delta_{\mathfrak{h}_*} + \gamma_{\mathfrak{k}_*}$ becomes maximal, so that for any $\iota \in \{\,1,\dots,l\,\}$

$$-\delta_{\mathfrak{h}_\iota} + \gamma_{\mathfrak{k}_\iota} \quad\le\quad -\delta_{\mathfrak{h}_*} + \gamma_{\mathfrak{k}_*}. \tag{3}$$

Let i' be minimal with $v_{i'} \in V^{\delta_{\mathfrak{h}_*}}$ and j' be minimal with $w_{j'} \in W^{\gamma_{\mathfrak{k}_*}}$. Next, write $(\underline{\delta},\underline{\gamma}) = (\underline{\delta}_1,\underline{\gamma}_1) + (\underline{\delta}_2,\underline{\gamma}_2)$. Here, $\underline{\delta}_1 = \sum_{i=1}^{i'-1} \alpha_i \delta^{(i)}$, $\underline{\gamma}_1 = \sum_{j=1}^{j'-1} \beta_j \gamma^{(j)}$, $\underline{\delta}_2 = \sum_{i=i'}^{p-1} \alpha_i \delta^{(i)}$, and $\underline{\gamma}_2 = \sum_{j=j'}^{q-1} \beta_j \gamma^{(j)}$. We begin by decomposing $(\underline{\delta}_2,\underline{\gamma}_2)$. First, let i'' be maximal with $v_{i''} \in V^{<\delta_{\mathfrak{h}_{(*+1)}}}$ and j'' maximal with $w_{j''} \in W^{<\gamma_{\mathfrak{k}_{(*+1)}}}$. Let $\underline{\delta}'_2 := \sum_{i=i'}^{i''} \alpha_i \delta^{(i)}$ and $\underline{\gamma}'_2 := \sum_{j=j'}^{j''} \beta_j \gamma^{(j)}$. Observe that, by (2), $f(V^{(i)}) \subset W^{(j)}$ for all i,j with $\alpha_i \cdot \beta_j \neq 0$ (the first non-zero coeffient of a $\gamma^{(j)}$ is $\beta_{\widetilde{j}}$ where $\widetilde{j}$ is maximal such that $w_{\widetilde{j}} \in W^{\le \gamma_{\mathfrak{k}_*}}$, by (1)). By Equation (1) and (3)

$$\sum_{i=i'}^{i''} p\alpha_i \quad=\quad \delta_{\mathfrak{h}_{(*+1)}} - \delta_{\mathfrak{h}_*} \quad\ge\quad \gamma_{\mathfrak{k}_{(*+1)}} - \gamma_{\mathfrak{k}_*} \quad=\quad \sum_{j=j'}^{j''} q\beta_j.$$

Thus, we can decompose

$$(\underline{\delta}'_2,\underline{\gamma}'_2) \quad=\quad \sum_{i=i'}^{i''} \alpha'_i \delta^{(i)} + \sum_{\substack{i=i',\dots,i'';\\ j=j',\dots,j''}} \eta_{i,j}\Big(\frac{1}{p}\delta^{(i)}, \frac{1}{q}\gamma^{(j)}\Big)$$

according to Lemma 3.6, i). Moreover, $\eta_{i,j} \neq 0$ implies $f(V^{(i)}) \subset W^{(j)}$. Next, define i''' as the maximal index with $v_{i'''} \in V^{<\delta_{\mathfrak{h}_{(*+2)}}}$ and let j''' be maximal with $w_{j'''} \in W^{<\gamma_{\mathfrak{k}_{(*+2)}}}$. Set $\underline{\delta}''_2 := \sum_{i=i'}^{i''} \alpha'_i \delta^{(i)} + \sum_{i=i''+1}^{i'''} \alpha_i \delta^{(i)}$, and $\underline{\gamma}''_2 = \sum_{j=j''+1}^{j'''} \beta_j \gamma^{(j)}$. Since, for the same reason as before, $\sum_{i=i'}^{i'''} p\alpha_i \ge \sum_{j=j'}^{j'''} q\beta_j$, it is clear that $\sum_{i=i'}^{i''} p\alpha'_i + \sum_{i=i''+1}^{i'''} p\alpha_i \ge \sum_{j=j''+1}^{j'''} q\beta_j$. Again, $f(V^{(i)}) \subset W^{(j)}$ for every i,j, such that either $\alpha'_i\beta_j \neq 0$ or $\alpha_i\beta_j \neq 0$. Hence, we can apply Lemma 3.6 again.

Now, iterate this process until all the β_j's of the beginning with $j < j_0$ are eaten up. The result is a decomposition

$$(\underline{\delta}_2,\underline{\gamma}_2) \quad=\quad \sum_{i=i'}^{p-1} \widetilde{\alpha}_i \delta^{(i)} + \sum_{\substack{i=i',\dots,p-1;\\ j=j',\dots,j_0-1}} \eta_{i,j}\Big(\frac{1}{p}\delta^{(i)}, \frac{1}{q}\gamma^{(j)}\Big) + \sum_{j=j_0}^{q-1} \beta_j \gamma^{(j)},$$

such that $\eta_{i,j} \neq 0$ implies $f(V^{(i)}) \subset W^{(j)}$. The corresponding decomposition of the remaining vector

$$(\underline{\delta}_1, \underline{\gamma}_1) \quad = \quad \sum_{i=1}^{i'_0-1} \alpha_i \delta^{(i)} + \sum_{\substack{i=i'_0,\dots,i'-1;\\ j=1,\dots,j'-1}} \eta_{i,j} \Big(\frac{1}{p} \delta^{(i)}, \frac{1}{q} \gamma^{(j)} \Big) + \sum_{j=1}^{j'-1} \widetilde{\beta}_j \gamma^{(j)}$$

is achieved by an analogous method, this time "working backwards" and making use of (3) and Lemma 3.6, ii). By definition of G, we find indices i_* and j_* such that $v_{i_*} \in V^{\delta_{\mathfrak{h}_*}}$, $w_{j_*} \in W^{\gamma_{\mathfrak{k}_*}}$, and $f_{i_*, j_*} \neq 0$. This completes the proof.

The general case

Let $([w_i], i \in V; [f_a], a \in A) \in \mathbb{P}_{\underline{r}, \underline{\mathrm{P}}}$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ be a one parameter subgroup of $\mathrm{SL}(\underline{\mathrm{P}})$. Choose bases $v^i_1, \dots, v^i_{p_i}$ for V_i, $i = 1, \dots, n$, such that λ is given w.r.t. those bases by the weight vector $\underline{\gamma}$, and let $Q(\lambda)$ be the full subquiver associated with the set $V(\lambda) := \{ i \in V \,|\, \lambda_i \neq \lambda_i^{(0)} \}$. Using Theorem 3.4, for each arrow $a \in A(\lambda)$, we pick $f^a_{i(t(a))_* i(h(a))_*} \neq 0$ (in the corresponding decomposition of f_a) and a decomposition of the weight vector $(\underline{\gamma}^{t(a)}, \underline{\gamma}^{h(a)})$ as

$$\sum_{\substack{i(t(a))=0,\dots,p_{t(a)}-1;\\ i(h(a))=0,\dots,p_{h(a)}-1}} \eta^0_{i(t(a)) i(h(a))} \Big(\frac{1}{p_{t(a)}} \gamma^{t(a),(i(t(a)))}, \frac{1}{p_{h(a)}} \gamma^{h(a),(i(h(a)))} \Big)$$

with the properties asserted in Theorem 3.4. Next, suppose we are given a decomposition $\underline{\gamma} = \sum_{\underline{j}} \eta_{\underline{j}} \gamma^{\underline{j}}$. This yields, for every arrow $a \in Q(\lambda)$, a decomposition of $(\underline{\gamma}^{t(a)}, \underline{\gamma}^{h(a)})$ as

$$\sum_{\substack{i(t(a))=0,\dots,p_{t(a)}-1;\\ i(h(a))=0,\dots,p_{h(a)}-1}} \eta_{i(t(a)) i(h(a))} \Big(\frac{1}{p_{t(a)}} \gamma^{t(a),(i(t(a)))}, \frac{1}{p_{h(a)}} \gamma^{h(a),(i(h(a)))} \Big)$$

with

$$\eta_{i(t(a)) i(h(a))} := \sum_{\substack{(j_1,\dots,j_n): j_{t(a)}=i(t(a));\\ j_{h(a)}=i(h(a))}} \eta_{\underline{j}}.$$

Theorem 3.7. *There exists a decomposition $\underline{\gamma} = \sum_{\underline{j}} \eta_{\underline{j}} \gamma^{\underline{j}}$ into basic weight vectors with the property that, for every arrow $a \in A$, $i(t(a)) = 0, \dots, p_{t(a)} - 1$, and $i(h(a)) = 0, \dots, p_{h(a)} - 1$*

$$\sum_{\substack{(j_1,\dots,j_n): j_{t(a)}=i(t(a));\\ j_{h(a)}=i(h(a))}} \eta_{\underline{j}} \quad = \quad \eta^0_{i(t(a)) i(h(a))}.$$

Proof. This will be done by induction, the case $\#V = 2$ being already settled. In a suitable labelling of the vertices, $\mathrm{Star}_Q(n) = \{ n-1, n \}$. Let us assume for simplicity that the arrow in $\mathrm{Star}_Q(n)$ is $(n-1, n)$. By hypothesis, there exist decompositions

$(\underline{\gamma}^1,\dots,\underline{\gamma}^{n-1}) = \sum_{\underline{j}=(j_1,\dots,j_{n-1},0)} \widetilde{\eta}_{\underline{j}}\underline{\gamma}^{\underline{j}}$ and $(0,\dots,0,\underline{\gamma}^{n-1},\underline{\gamma}^n) = \sum \eta^0_{i(n-1)i(n)}\gamma^{(0,\dots,0,i(n-1),i(n))}$ with the respective properties. Since, by (1), for each $i(n-1) = 1,\dots,p_{n-1}-1$,

$$\sum_{i(n)} \eta^0_{i(n-1)i(n)} \quad = \quad \sum_{\substack{(j_1,\dots,j_{n-1},0):\\ j_{n-1}=i(n-1)}} \widetilde{\eta}_{\underline{j}},$$

the assertion is obvious.

Corollary 3.8. *A point $([w_i], i \in V; [f_a], a \in A)$ in $\mathbb{P}_{\underline{\mathfrak{r}},\underline{\mathrm{p}}}$ is (semi)stable w.r.t. chosen the linearization in $\mathscr{O}(l_1,\dots,l_n; b_a, a \in A)$, if and only if the Hilbert-Mumford criterion is fulfilled for all basic one parameter subgroups.*

To conclude, let us look at the weigths of the basic one parameter subgroups. In the above notation, let $\underline{\gamma}^{\underline{j}}$ be a basic weight vector, and $Q' := Q(\lambda^{\underline{j}})$. Then,

$$\begin{aligned} \mu\Big(([f_a], a \in A), \lambda^{\underline{j}}\Big) \quad = \quad & \sum_{a \in A'} b_a\Big(-\frac{j_{t(a)}}{p_{t(a)}} + \frac{j_{h(a)}}{p_{h(a)}}\Big) \\ & + \sum_{i \in \mathrm{END}_Q(Q'); a \in \mathrm{Out}_Q(i)} b_a\Big(-\frac{j_{t(a)}}{p_{t(a)}} + \varepsilon_a(V^{(j_{t(a)})}_{t(a)})\Big) \\ & + \sum_{i \in \mathrm{END}_Q(Q'); a \in \mathrm{In}_Q(i)} b_a\Big(\frac{j_{h(a)}}{p_{h(a)}} - \varepsilon_a(V^{(j_{h(a)})}_{h(a)})\Big). \end{aligned}$$

Here, $\varepsilon_a(V^{(j_{t(a)})}_{t(a)}) = 1$ if $V^{(j_{t(a)})}_{t(a)} \not\subset \ker f_a$ and 0 otherwise, and $\varepsilon_a(V^{(j_{h(a)})}_{h(a)}) = 1$ if $V^{(j_{h(a)})}_{h(a)} \supset \mathrm{Im}\, f_a$ and 0 otherwise. One easily sees:

Corollary 3.9. *Given bases $v^i_1,\dots,v^i_{p_i}$ for V_i, $i \in V$, and indices $j_i \in \{0,\dots,p_i\}$ with*

$$f_a\big(V^{(j_{t(a)})}_{t(a)}\big) \quad \subset \quad V^{(j_{h(a)})}_{h(a)}, \qquad \textit{for all } a \in A,$$

one has

$$\mu\Big(([f_a], a \in A), \lambda^{(j_1,\dots,j_n)}\Big) \quad \le \quad \sum_{a \in A} b_a\Big(-\frac{j_{t(a)}}{p_{t(a)}} + \frac{j_{h(a)}}{p_{h(a)}}\Big)$$

with equality in case $\underline{\gamma}^{(j_1,\dots,j_n)}$ is basic w.r.t. $([f_a], a \in A)$.

Proof of Theorem 3.3

If we assume that $l_i = \sum_{a \in A(i)} b_a \alpha_{i,a}$ for some positive rational numbers $\alpha_{i,a}$, we find in the setting of Corollary 3.9

$$\begin{aligned} & \mu\Big(\big([w_i], i \in V; [f_a], a \in A\big), \lambda^{(j_1,\dots,j_n)}\Big) \\ \le \quad & \sum_{a \in A} b_a\Big[\alpha_{t(a),a}\frac{1}{p_{t(a)}}\mu\big([w_{t(a)}], \lambda^{(j_{t(a)})}_{t(a)}\big) - \frac{j_{t(a)}}{p_{t(a)}} \\ & \qquad + \alpha_{h(a),a}\frac{1}{p_{h(a)}}\mu\big([w_{h(a)}], \lambda^{(j_{h(a)})}_{h(a)}\big) + \frac{j_{h(a)}}{p_{h(a)}}\Big] \end{aligned}$$

with equality in case $\underline{\gamma}^{(j_1,\dots,j_n)}$ is basic w.r.t. $([f_a], a \in A)$. The assertion now follows from Corollary 3.8.

Remark 3.10. Observe that we made use of the Additivity Assumption 3.1 in this computation. It is not valid without it.

4 Proof of Theorem 1.6

We can now start with the GIT construction which follows a well-known pattern. For the case of semistable torsion free coherent sheaves, one may consult [7]. The details left out here can be easily filled in with that reference.

Boundedness

We must first convince ourselves that the semistable objects live in bounded families, i.e., can be parametrized by a scheme of finite type over $\mathbb{C}$. Here, we check that this is true for the participating torsion free coherent sheaves.

The data $\underline{P}$, $\underline{\sigma}_Q$, and $\underline{b}_Q$ are the same as before, and $\vartheta := \vartheta(\underline{P}, \underline{\sigma}_Q, \underline{b}_Q)$. Let $\underline{R} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ be a ϑ-semistable representation of type $\underline{P}$, and $i_0 \in V$ a fixed vertex. Observe that Γ_Q's being a tree implies that every other vertex can be connected to i_0 by a unique (shortest) path. For any non-trivial, proper subsheaf $\mathscr{G}$ of $\mathscr{E}_{i_0}$, we define the subrepresentation $\underline{R}_{\mathscr{G}} = (\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ of $\underline{R}$ as follows: We set $\mathscr{F}_{i_0} := \mathscr{G}$, and for any other vertex i, we define $\mathscr{F}_i$ as 0 if the path connecting i to i_0 passes through an ingoing arrow of i_0 and as $\mathscr{E}_i$ in the other case, and finally, $\varphi'_a := \varphi_{a|\mathscr{F}_{t(a)}}$ for all $a \in A$. The condition of ϑ-semistability applied to $\underline{R}_{\mathscr{G}}$ shows that

$$\begin{aligned} &\textstyle\sum_{a\in \mathrm{Out}_Q(i_0)} \quad b_a\left[\check{\sigma}_{i_0}\left\{P(\mathscr{F}_{i_0}) - \mathrm{rk}\,\mathscr{F}_{i_0}\left(\frac{P(\mathscr{E}_{i_0}) - \sigma_{i_0}}{\mathrm{rk}\,\mathscr{E}_{i_0}} + \frac{\sigma_{i_0}}{\mathrm{rk}\,\mathscr{F}_{i_0}}\right)\right\}\right] \\ &+\textstyle\sum_{a\in \mathrm{In}_Q(i_0)} \quad b_a\left[\check{\sigma}_{i_0}\left\{P(\mathscr{F}_{i_0}) - \mathrm{rk}\,\mathscr{F}_{i_0}\frac{P(\mathscr{E}_{i_0}) + \sigma_{i_0}}{\mathrm{rk}\,\mathscr{E}_{i_0}}\right\}\right] \end{aligned}$$

is a non-positive polynomial, whence

$$\Big(\sum_{a\in A(i_0)} b_a\check{\sigma}_{i_0}\Big)\frac{P(\mathscr{F}_{i_0})}{\mathrm{rk}\,\mathscr{F}_{i_0}} \quad \le \quad \Big(\sum_{a\in A(i_0)} b_a\check{\sigma}_{i_0}\Big)\frac{P_{i_0}}{r_{i_0}} + \sum_{a\in A(i_0)} b_a\sigma. \tag{4}$$

Let $\overline{\sigma}^{\vee}_{i_0}$ be the leading coefficient of the polynomial $\sum_{a\in A(i_0)} b_a\check{\sigma}_{i_0}$, $\check{s}_{i_0}$ its degree, $\overline{\sigma}_{i_0}$ the coefficient of the term of degree $(\check{s}_{i_0} + \dim X - 1)$ in $(\sum_{a\in A} b_a)\sigma$ (the latter polynomial has degree at most $(\check{s}_{i_0} + \dim X - 1)$), and define the constant $C_{i_0} := \overline{\sigma}_{i_0}/\overline{\sigma}^{\vee}_{i_0}$. Thus, taking leading coefficients in (4) shows

$$\mu_{\max}(\mathscr{E}_{i_0}) \quad \le \quad \mu_{i_0} + C_{i_0},$$

whence, by invoking the boundedness theorem of Maruyama [7], Theorem 3.3.7, the following result is obtained:

Theorem 4.1. *For every $i \in V$, the set of isomorphy classes of torsion free coherent sheaves $\mathscr{E}_i$ with Hilbert polynomial $\underline{P}(i)$ showing up in ϑ-semistable representations of type $\underline{P}$ is bounded.*

Sectional semistability

Sectional semistability is a technical way of rewriting the semistability condition. It is the form in which we will encounter it during the GIT construction.

This time, we fix in addition to $\underline{P}$ and $\underline{b}_Q$ positive rational numbers s_i, $i \in V$, and set $s := s_1 \cdot \ldots \cdot s_n$, $\check{s}_i = s/s_i$, $i \in V$. A representation $(\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ of Q is called $(\underline{s}_Q, \underline{b}_Q)$-*sectional (semi)stable*, if there are subspaces $H_i \subset H^0(\mathscr{E}_i)$ of dimension $\chi(\mathscr{E}_i)$, $i \in V$, with $f_a(H_{t(a)}) \subset H_{h(a)}$, for all $a \in A$, such that, for all sub-representations $\underline{\mathscr{F}} = (\mathscr{F}_i, i \in V, \varphi'_a, a \in A)$, the number

$$\begin{aligned} \delta(\underline{\mathscr{F}}, \underline{\mathscr{E}}, \underline{\varphi}) \quad := \quad & \sum_{a \in A} b_a \Bigg[\check{s}_{t(a)} \left\{ \dim\big(H_{t(a)} \cap H^0(\mathscr{F}_{t(a)})\big) - \operatorname{rk} \mathscr{F}_{t(a)} \frac{\chi(\mathscr{E}_{t(a)}) - s_{t(a)}}{\operatorname{rk} \mathscr{E}_{t(a)}} \right\} \\ & + \quad \check{s}_{h(a)} \left\{ \dim\big(H_{h(a)} \cap H^0(\mathscr{F}_{h(a)})\big) - \operatorname{rk} \mathscr{F}_{h(a)} \frac{\chi(\mathscr{E}_{h(a)}) + s_{h(a)}}{\operatorname{rk} \mathscr{E}_{h(a)}} \right\} \Bigg] \end{aligned}$$

is negative (non-positive).

Theorem 4.2. *There exists an m_0, such that, for all $i_0 \in V$, the set of isomorphy classes of torsion free coherent sheaves $\mathscr{E}_{i_0}$ occuring in representations $(\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ of type $\underline{P}$ with the property that there exists an $m \geq m_0$ such that $(\mathscr{E}_i(m), i \in V; \varphi_a(m), a \in A)$ is sectional semistable w.r.t. the parameters $(\sigma_i(m), i \in V; b_a, a \in A)$ is bounded, too.*

Proof. Before starting with the proof, we remind the reader of the following important result.

Theorem (The Le Potier-Simpson estimate [7], p.71**).** *Let $\mathscr{F}$ be a torsion free coherent sheaf, and $C(\mathscr{F}) := \operatorname{rk}\mathscr{F}(\operatorname{rk}\mathscr{F} + \dim X)/2$. Then, for every $m \geq 0$,*

$$\begin{aligned} \frac{h^0(\mathscr{F}(m))}{\operatorname{rk}\mathscr{F}} \quad \leq \quad & \frac{\operatorname{rk}\mathscr{F} - 1}{(\dim X)!\operatorname{rk}\mathscr{F}} \big[\mu_{\max}(\mathscr{F}) + C(\mathscr{F}) - 1 + m\big]_+^{\dim X} \\ & + \frac{1}{(\dim X)!\operatorname{rk}\mathscr{F}} \big[\mu(\mathscr{F}) + C(\mathscr{F}) - 1 + m\big]_+^{\dim X}. \end{aligned}$$

As in the proof of Theorem 4.1, for $i_0 \in V$ and a non-trivial subsheaf $\mathscr{G}$ of $\mathscr{E}_{i_0}$, we find, for $m \gg 0$, an upper bound

$$\frac{h^0(\mathscr{G}(m))}{\operatorname{rk}\mathscr{G}} \quad \leq \quad \frac{\chi(\mathscr{E}_{i_0}(m))}{\operatorname{rk}\mathscr{E}_{i_0}} + K_{i_0}(m) \quad \leq \quad \frac{h^0(\mathscr{E}_{i_0}(m))}{\operatorname{rk}\mathscr{E}_{i_0}} + K_{i_0}(m).$$

Here, $K_{i_0}(m)$ is a positive rational function growing at most like a polynomial of degree $\dim X - 1$ depending only on the input data $\underline{\sigma}_Q$, $\underline{b}_Q$, and $\underline{P}$. Hence, for every non-trivial quotient $\mathscr{Q}$ of $\mathscr{E}_{i_0}$, we get a lower bound

$$\begin{aligned} \frac{\chi(\mathscr{E}_{i_0}(m))}{\operatorname{rk}\mathscr{E}_{i_0}} - (\operatorname{rk}\mathscr{E}_{i_0} - 1)K_{i_0}(m) \quad \leq \quad & \frac{h^0(\mathscr{E}_{i_0}(m))}{\operatorname{rk}\mathscr{E}_{i_0}} - (\operatorname{rk}\mathscr{E}_{i_0} - 1)K_{i_0}(m) \\ \leq \quad & \frac{h^0(\mathscr{Q}(m))}{\operatorname{rk}\mathscr{Q}}. \end{aligned}$$

As remarked before, the left hand side is positive for all $m \gg 0$ and depends only on the input data $\underline{\sigma}_Q$, $\underline{b}_Q$, and $\underline{\mathrm{P}}$, and we can bound K_{i_0} from above by a polynomial $k_{i_0}(m)$ of degree at most $\dim X - 1$. Thus, applying the above estimate to the minimal slope destabilizing quotient sheaf of $\mathscr{E}_{i_0}$ and making use of the Le Potier-Simpson estimate, we find

$$\frac{P_{i_0}(m)}{r_{i_0}} - (r_{i_0} - 1)k_{i_0}(m) \quad \leq \quad \frac{1}{\dim X!}\big[\mu_{\min}(\mathscr{E}_{i_0}) + C - 1 + m\big]_+^{\dim X}, \tag{5}$$

$C := \max\{\, s(s+\dim X)/2 \,|\, s = 1, \ldots, r_{i_0} - 1\,\}$. Let D be the coefficient of $m^{\dim X - 1}$ in the left hand polynomial multiplied by $(\dim X)!$. Then, we can find an m_{i_0} such that for all $m \geq m_{i_0}$,

$$\frac{P_{i_0}(m)}{r_{i_0}} - (r_{i_0} - 1)k_{i_0}(m) \quad > \quad \frac{1}{\dim X!}\big[D - \frac{1}{r!} + m\big]_+^{\dim X}.$$

Thus, from (5) and the assumption that there exist an $m \geq m_{i_0}$ such that $(\mathscr{E}_i(m), i \in V; \varphi_a(m), a \in A)$ is sectional semistable w.r.t. the parameters $(\sigma_i(m), i \in V; b_a, a \in A)$, one finds the lower bound $\mu_{\min}(\mathscr{E}_{i_0}) \geq D - C + 1$ and, consequently, an upper bound for $\mu_{\max}(\mathscr{E}_{i_0})$, so that the theorem follows again from Maruyama's boundedness theorem.

Our next task will be to describe the relation between sectional semistability for large m and ϑ-semistability. For this, we have to remind the reader of the following:

Let $\mathscr{E}$ be a torsion free coherent sheaf. A subsheaf $\mathscr{F} \subset \mathscr{E}$ is called *saturated*, if $\mathscr{E}/\mathscr{F}$ is again torsion free. The sheaf $\widetilde{\mathscr{F}} := \ker\big(\mathscr{E} \longrightarrow (\mathscr{E}/\mathscr{F})/\mathrm{Tors}(\mathscr{E}/\mathscr{F})\big)$ is called the *saturation of* $\mathscr{F}$. It has the same rank as $\mathscr{F}$ and $P(\mathscr{F}) \leq P(\widetilde{\mathscr{F}})$. It is often necessary to restrict to saturated subsheaves, because of the following boundedness result:

Theorem. *Let $\mathfrak{C}$ be a bounded family of torsion free coherent sheaves on X. Given a constant C, the set of isomorphism classes of sheaves $\mathscr{F}$ for which there exist a sheaf $\mathscr{E}$ with $[\mathscr{E}] \in \mathfrak{C}$ and a saturated subsheaf $\mathscr{F}'$ of $\mathscr{E}$ with $C \leq \mu(\mathscr{F}')$ and $\mathscr{F} \cong \mathscr{F}'$ is also bounded.*

Proof. Note that one can find a finite dimensional vector space U and an integer m, such that every sheaf $\mathscr{E}$ with $[\mathscr{E}] \in \mathfrak{C}$ can be written as a quotient of $U \otimes \mathscr{O}_X(-m)$. Our assumption yields a lower bound $\mu(\mathscr{E}/\mathscr{F}') \geq C'$ for some C' which can be computed from C and the maximal slope occuring for a sheaf in $\mathfrak{C}$. Grothendieck's lemma ([7], Lem. 1.7.9) shows that the family of sheaves of the form $\mathscr{E}/\mathscr{F}'$ (viewed as a quotient of $U \otimes \mathscr{O}_X(-m)$) is bounded, whence also the family of sheaves of the form $\mathscr{F}'$.

Now, let $\underline{\mathrm{R}} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ be a representation of Q and $\underline{\mathrm{R}}' := (\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ a sub-representation. Then, for every arrow $a \in A$, $\varphi_a(\widetilde{\mathscr{F}}_{t(a)}) \subset \widetilde{\mathscr{F}}_{h(a)}$, so that $\widetilde{\underline{\mathrm{R}}}' := (\widetilde{\mathscr{F}}_i, i \in V; \varphi_{a|\widetilde{\mathscr{F}}_{t(a)}}, a \in A)$ is another sub-representation of $\underline{\mathrm{R}}$ with $\vartheta(\underline{\mathrm{R}}') \leq \vartheta(\widetilde{\underline{\mathrm{R}}}')$. Therefore, ϑ-(semi)stability needs to be checked only for *saturated sub-representations* $\underline{\mathrm{R}}'$, i.e., sub-representations where $\mathscr{F}_i$ is a saturated subsheaf of $\mathscr{E}_i$, $i \in V$.

Proposition 4.3. *There exists a number $m_1 \geq m_0$, such that for every $m \geq m_0$, a representation $(\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ of type $\underline{\mathrm{P}}$ is ϑ-(semi)stable, if and only if $(\mathscr{E}_i(m), i \in V; \varphi_a(m), a \in A)$ is sectional (semi)stable w.r.t. the parameters $(\sigma_i(m), i \in V; b_a, a \in A)$.*

Proof. We show the "only if"-direction and take $H_i = H^0(\mathscr{E}_i(m))$. The other direction is similar but easier, and, hence, omitted. First, let $\mathscr{E}$ be an arbitary torsion free coherent sheaf of slope μ and rank r, and suppose we are given a bound $\mu_{\max}(\mathscr{E}) \leq \mu + K$, for some constant K. Let $\mathscr{F}$ be a subsheaf with $\mu(\mathscr{F}) \leq \mu - rC(\mathscr{E}) - (r-1)K - K'$, for some other constant K'. Then, the Le Potier-Simpson estimate tells us that for large m

$$\frac{h^0(\mathscr{F}(m))}{\operatorname{rk}\mathscr{F}} \quad \leq \quad \frac{m^{\dim X}}{\dim X!} + \frac{m^{\dim X - 1}}{(\dim X - 1)!}(\mu - K - 1) + \text{lower order terms.}$$

Now, suppose we are given some constants K_i, $i \in V$ (not to be confounded with the rational functions of the same name we have considered before). Then, by the above, for every sub-representation $(\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ with $\mu(\mathscr{F}_i) \leq \mu_i - r_i C(\mathscr{E}_i) - (r_i - 1)C_i - K_i$, the C_i being the constants obtained in the proof of Theorem 4.1, we get an estimate

$$\delta\Big(\underline{\mathscr{F}}(m), \underline{\mathscr{E}}(m), \underline{\varphi}(m)\Big) \quad \leq \quad \delta\big(\underline{\sigma}_Q, \underline{b}_Q, P_i, K_i, i \in V\big)$$

where the right hand side is a polynomial of degree at most $\dim X - 1$ depending only on the data in the bracket. Due to the fact that the σ_i, $i \in V$, may have different degrees, that polynomial might not be homogeneous. First, we define $V_1 := \{\, i \in V \mid \check{\sigma}_i \text{ has maximal degree} \,\}$. Given the fact that we have already uniformly bounded $\mu_{\max}(\mathscr{E}_i)$ for all $i \in V$, it is now clear that we can choose the constants K_i, $i \in V_1$, in such a way that for every sub-representation $(\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ with $\mu(\mathscr{F}_{i_0}) \leq \mu_{i_0} - r_{i_0} C(\mathscr{E}_{i_0}) - (r_{i_0} - 1)C_{i_0} - K_{i_0}$ for *one single* index $i_0 \in V_1$, the function $\delta(\underline{\mathscr{F}}(m), \underline{\mathscr{E}}(m), \underline{\varphi}(m))$ will be negative for all large m. Hence, we may restrict our attention to saturated sub-representations $(\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ where $\mu(\mathscr{F}_i) > \mu_i - r_i C(\mathscr{E}_i) - (r_i - 1)C_i - K_i$ for all $i \in V_1$. But this means that the $\mathscr{F}_i$, $i \in V_1$, vary in bounded families, whence they can all be assumed to be globally generated and without higher cohomology. Under these circumstances, the fact that $(\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ is ϑ-semistable implies that the sum of the contributions to $\delta(\underline{\mathscr{F}}(m), \underline{\mathscr{E}}(m), \underline{\varphi}(m))$ of the terms coming from σ_i's with $i \in V_1$ cannot be positive for large m. Therefore, we look at $V_2 := \{\, i \in V \mid \check{\sigma}_i \text{ has second largest degree} \,\}$. By iterating the above procedure, we finally get the claim.

The above argumentation shows that the sub-representations of interest live in bounded families, whence we conclude

Theorem 4.4. *There is a number $m_2 \geq m_1$, such that for all $m \geq m_2$ the following conditions on a representation $\underline{R} = (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ of type $\underline{P}$ are equivalent.*

1. *$\underline{R}$ is ϑ-(semi)stable.*
2. *$(\mathscr{E}_i(m), i \in V; \varphi_a(m), a \in A)$ fulfills the condition $(\sigma_i(m), i \in V; \underline{b}_Q)$-sectional (semi) stabililty for all sub-representations $(\mathscr{F}_i(m), i \in V; \varphi'_a(m), a \in A)$, such that $\mathscr{F}_i(m)$ is globally generated for all $i \in V$.*

The parameter space

By Theorem 4.1, we can find a natural number m_3 which we choose larger than m_2 in Theorem 4.4, such that for every $m \geq m_3$ and every ϑ-semistable representation $(\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ of type $\underline{P}$, the sheaves $\mathscr{E}_i(m)$ are globally generated and without higher cohomology, $i \in V$. Fix such an m, set $p_i := P_i(m)$, and choose complex vector spaces V_i of

dimension p_i, $i \in V$. For every $i \in V$, let $\mathfrak{Q}_i'$ be the quasi-projective quot scheme of equivalence classes of quotients $k: V_i \otimes \mathscr{O}_X(-m) \longrightarrow \mathscr{E}_i$ where $\mathscr{E}_i$ is a torsion free coherent sheaf with Hilbert polynomial P_i and $H^0(k(m))$ is an isomorphism, and define $\mathfrak{Q}_i$ as the union of those components of $\mathfrak{Q}_i'$ which contain members of ϑ-semistable representations.

For a point $([k_i: V_i \otimes \mathscr{O}_X(-m) \longrightarrow \mathscr{E}_i, i \in V]) \in \prod_{i \in V} \mathfrak{Q}_i$ and an arrow $a \in A$, an element $\varphi_a \in \mathrm{Hom}(\mathscr{E}_{t(a)}, \mathscr{E}_{h(a)})$ is uniquely determined by the associated homomorphism $f_a := H^0(k_{h(a)}(m))^{-1} \circ H^0(\varphi_a(m)) \circ H^0(k_{t(a)}(m)) \in \mathrm{Hom}(V_{t(a)}, V_{h(a)})$. Define the spaces $\mathfrak{P} := \prod_{a \in A} \mathbb{P}(\mathrm{Hom}(V_{t(a)}, V_{h(a)})^\vee)$, $\mathfrak{U} := (\prod_{i \in V} \mathfrak{Q}_i) \times \mathfrak{P}$, denote by $\mathfrak{N}_a$ the pullback of the sheaf $\mathscr{O}_{\mathbb{P}(\mathrm{Hom}(V_{t(a)}, V_{h(a)})^\vee)}(1)$ to $\mathfrak{U} \times X$, and let

$$\mathfrak{f}_a: V_{t(a)} \otimes \mathscr{O}_{\mathfrak{U} \times X} \longrightarrow V_{h(a)} \otimes \mathfrak{N}_a$$

be the pullback of the universal homomorphism on $\mathbb{P}(\mathrm{Hom}(V_{t(a)}, V_{h(a)})^\vee)$ to $\mathfrak{U} \times X$. Moreover, let $\widetilde{\mathfrak{k}}_i: V_i \otimes \mathscr{O}_{\mathfrak{U} \times X} \longrightarrow \widetilde{\mathfrak{E}}_i$ be the pullbacks of the universal quotients on $\mathfrak{Q}_i \times X$ twisted by $\mathrm{id}_{\pi_X^* \mathscr{O}_X(m)}$, $i \in V$. Composing $\mathfrak{f}_a$ and $\widetilde{\mathfrak{k}}_{h(a)} \otimes \mathfrak{N}_a$, we obtain a homomorphism

$$\widetilde{\mathfrak{f}}_a: V_{t(a)} \otimes \mathscr{O}_{\mathfrak{U} \times X} \longrightarrow \widetilde{\mathfrak{E}}_{h(a)} \otimes \mathfrak{N}_a.$$

We define $\mathfrak{T}$ as the closed subscheme whose closed points are those $u \in \mathfrak{U}$ for which $\widetilde{\mathfrak{f}}_{a|\{u\} \times X}$ vanishes on $\ker(\widetilde{\mathfrak{k}}_{t(a)|\{u\} \times X})$ for all $a \in A$, i.e., those u for which the restriction $\widetilde{\mathfrak{f}}_{a|\{u\} \times X}$ comes from a homomorphism $\varphi_{u,a}: \widetilde{\mathfrak{E}}_{t(a)|\{u\} \times X}(-m) \longrightarrow \widetilde{\mathfrak{E}}_{h(a)|\{u\} \times X}(-m)$. The scheme $\mathfrak{T}$ is the common vanishing locus of the vector bundle maps

$$\pi_{\mathfrak{U}*}\Big(\ker \widetilde{\mathfrak{k}}_{t(a)} \otimes \pi_X^* \mathscr{O}_X(l)\Big) \longrightarrow \pi_{\mathfrak{U}*}\Big(\widetilde{\mathfrak{E}}_{h(a)} \otimes \mathfrak{N}_a \otimes \pi_X^* \mathscr{O}_X(l)\Big), \quad a \in A,$$

obtained by projecting $\widetilde{\mathfrak{f}}_a \otimes \pi_X^* \mathscr{O}_X(l)$ for l large enough. Set $\mathfrak{E}_{\mathfrak{T},i} := \widetilde{\mathfrak{E}}_{i|\mathfrak{T} \times X}$, $i \in V$, and let $\varphi_{\mathfrak{T},a}: \mathfrak{E}_{\mathfrak{T},t(a)} \longrightarrow \mathfrak{E}_{\mathfrak{T},h(a)} \otimes \mathfrak{N}_{a|\mathfrak{T} \times X}$, $a \in A$, be the induced homomorphisms. The family $(\mathfrak{E}_{\mathfrak{T},i}, i \in V; \varphi_{\mathfrak{T},a}, a \in A)$ will be abusively called the *universal family* though it is not exactly a family in the sense of our definition.

Recall that, for each $i \in V$, there is a left action of $\mathrm{SL}(V_i)$ on $\mathfrak{Q}_i$ given on closed points by

$$g \cdot [k: V_i \otimes \mathscr{O}_X(-m) \longrightarrow \mathscr{E}_i] \quad = \quad [V_i \otimes \mathscr{O}_X(-m) \stackrel{g^{-1} \otimes \mathrm{id}}{\longrightarrow} V_i \otimes \mathscr{O}_X(-m) \stackrel{k}{\longrightarrow} \mathscr{E}_i].$$

Thus, there is an induced left action of $G := \prod_{i \in V} \mathrm{SL}(V_i)$ on $(\prod_{i \in V} \mathfrak{Q}_i) \times \mathfrak{P}$ which leaves the parameter space $\mathfrak{T}$ invariant and thus yields an action

$$\alpha: G \times \mathfrak{T} \longrightarrow \mathfrak{T}.$$

The proof of the following is standard (cf., e.g., [16]) and, therefore, will not be given here.

Proposition 4.5. i) *The space $\mathfrak{T}$ enjoys the* local universal property*, i.e., for every noetherian scheme S and every family $(\underline{\mathfrak{E}}_S, \underline{\varphi}_S) = (\mathfrak{E}_{S,i}, i \in V; \varphi_{S,a}, a \in A)$ of ϑ-semistable representations of type $\underline{P}$ parametrized by S, there exists an open covering $U_1, \ldots, U_s$ of S and morphisms $h_\nu: U_\nu \longrightarrow \mathfrak{T}$, such that the pullback of the universal family on $\mathfrak{T} \times X$ via $(h_\nu \times \mathrm{id}_X)$ is equivalent to $(\underline{\mathfrak{E}}_{S|U_\nu}, \underline{\varphi}_{S|U_\nu})$, $\nu = 1, \ldots, s$.*

ii) *Given a noetherian scheme S and two morphisms h^1, h^2, such that the pullbacks of $\mathfrak{N}_a$, $a\in A$, via $(h^1\times \mathrm{id}_X)$ and $(h^2\times \mathrm{id}_X)$ are trivial and the pullbacks of the universal family via $(h^1\times \mathrm{id}_X)$ and $(h^2\times \mathrm{id}_X)$ are equivalent, there exists an étale covering $\eta\colon T\longrightarrow S$ and a morphism $t\colon T\longrightarrow G$ with $(h^1\circ\eta)=t\cdot(h^2\circ\eta)$.*

As we will see below, the set $\mathfrak{T}_{\underline{\mathrm{P}}}^{\vartheta-(s)s}\subset\mathfrak{T}$ parametrizing ϑ-(semi)stable representations is open. For this reason and by the universal property of the good (geometric) quotient, i) and ii) of Theorem 1.6 are a direct consequence of the following

Theorem 4.6. *The good quotient $\mathfrak{T}_{\underline{\mathrm{P}}}^{\vartheta-ss}/\!\!/G$ exists as projective scheme, and the open subscheme $\mathfrak{T}_{\underline{\mathrm{P}}}^{\vartheta-s}/\!\!/G$ is a geometric quotient for $\mathfrak{T}_{\underline{\mathrm{P}}}^{\vartheta-s}$ w.r.t. the action of G.*

We have chosen to prove this result via Gieseker's method for the single reason that after determining the weights we are done and don't have to make any more considerations about polynomials and leading coefficients which, in our situation, would be rather messy, I guess.

The Gieseker space and the Gieseker map

For $i\in V$, let $\mathfrak{k}_i\colon V_i\otimes\pi_X^*\mathscr{O}_X(-m)\longrightarrow\mathfrak{E}_{\mathfrak{Q}_i}$ be the universal quotient on $\mathfrak{Q}_i\times X$. The line bundle $\det(\mathfrak{E}_{\mathfrak{Q}_i})$ induces a morphism $\mathfrak{d}_{\mathfrak{Q}_i}\colon\mathfrak{Q}_i\longrightarrow\operatorname{Pic}X$. Let $\mathfrak{A}_i$ be the union of the finitely many components of $\operatorname{Pic}X$ hit by this map. Note that this does not depend on the choice of m. Therefore, in addition to the other hypothesis' on m, we may assume that $\mathscr{L}(r_im)$ is globally generated and without higher cohomology, for every $[\mathscr{L}]\in\mathfrak{A}_i$, $i\in V$. Fix a Poincaré line bundle $\mathscr{N}$ on $\operatorname{Pic}X\times X$, let $\mathscr{N}_i$ be its restriction to $\mathfrak{A}_i\times X$, and set

$$\mathbb{G}_i\quad:=\quad\mathbb{P}\Big(\underline{\mathrm{Hom}}\big(\bigwedge^{r_i}V_i\otimes\mathscr{O}_{\mathfrak{A}_i},\pi_{\mathfrak{A}_i*}(\mathscr{N}_i\otimes\pi_X^*\mathscr{O}_X(r_im))\big)^{\vee}\Big).$$

The homomorphism

$$\pi_{\mathfrak{Q}_i*}\big(\overset{r_i}{\wedge}(\mathfrak{k}_i\otimes\mathrm{id}_{\pi_X^*\mathscr{O}_X(m)})\big)\colon\bigwedge^{r_i}V_i\otimes\mathscr{O}_{\mathfrak{Q}_i}\longrightarrow\pi_{\mathfrak{Q}_i*}\Big(\det(\mathfrak{E}_{\mathfrak{Q}_i})\otimes\pi_X^*\mathscr{O}_X(r_im)\Big)$$

gives rise to an injective and $\mathrm{SL}(V_i)$-equivariant morphism $\mathrm{Gies}_i\colon\mathfrak{Q}_i\longrightarrow\mathbb{G}_i$. Defining $\mathfrak{G}:=(\prod_{i\in V}\mathbb{G}_i)\times\mathfrak{P}$, we obtain from the above data a G-equivariant and injective morphism

$$\mathrm{Gies}\colon\mathfrak{T}\longrightarrow\mathfrak{G}.$$

We linearize the G-action on the right hand space in $\mathscr{O}(l_1,\dots,l_n;b_a,a\in A)$ where

$$l_i\quad:=\quad\sum_{a\in\mathrm{Out}_Q(i)}b_a\frac{p_{t(a)}-\sigma_{t(a)}(m)}{r_{t(a)}\sigma_{t(a)}(m)}+\sum_{a\in\mathrm{In}_Q(i)}b_a\frac{p_{h(a)}+\sigma_{h(a)}(m)}{r_{h(a)}\sigma_{h(a)}(m)}.$$

After these preparations, Theorem 4.6 and Theorem 1.6 will follow from

Theorem 4.7. i) *The image of a point $([k_i\colon V_i\otimes\mathscr{O}_X(-m)\longrightarrow\mathscr{E}_i],i\in V;[\varphi_a],a\in A)$ under* Gies *in $\mathfrak{G}$ is (semi/poly)stable w.r.t. the fixed linearization if and only if $(\mathscr{E}_i,i\in V_i;\varphi_a,a\in A)$ is a ϑ-(semi/poly)stable representation of Q of type $\underline{\mathrm{P}}$.*

ii) *The morphism $\mathrm{Gies}^{ss}\colon\mathfrak{T}_{\underline{\mathrm{P}}}^{\vartheta-ss}\longrightarrow\mathfrak{G}^{ss}$ is injective and proper and, therefore, finite.*

Proof. i) The Gieseker space $\mathfrak{G}$ maps G-invariantly to $\prod_i \mathfrak{A}_i$, and the fibres are closed, G-invariant subschemes, namely, over the point $\underline{\mathscr{L}} := (\mathscr{L}_1, ..., \mathscr{L}_n) \in \prod_i \mathfrak{A}_i$ sits the space

$$\mathfrak{G}_{\underline{\mathscr{L}}} \quad = \quad \prod_i \mathbb{P}\Big(\mathrm{Hom}\big(\bigwedge^{r_i} V_i, H^0(\mathscr{L}_i(r_i m))\big)^\vee\Big) \times \mathfrak{P}.$$

It suffices to determine the (semi)stable points in those spaces. In the following, we use the notation of Section 3. We need two formulas.

- Let $i \in V$ and $[k_i : V_i \otimes \mathscr{O}_X(-m) \longrightarrow \mathscr{E}_i]$ be a point in $\mathfrak{Q}_i$ with image $g_i := \mathrm{Gies}_i([k_i]) \in \mathbb{G}_i$. Given a basis $v^i_1, ..., v^i_{p_i}$ of V_i and an index $j_i \in \{0, ..., p_i\}$, we define

  $$\mathscr{E}_i^{(j)} \quad := \quad k_i\big(V_i^{(j)} \otimes \mathscr{O}_X(-m)\big).$$

  Recall that one has
  $$\mu\big(g_i, \lambda_i^{(j)}\big) \quad = \quad p_i \,\mathrm{rk}\, \mathscr{E}_i^{(j)} - j r_i \tag{6}$$
  and that Assumption 3.1 is verified on the subset $\mathrm{Gies}_i(\mathfrak{Q}_i) \subset \mathbb{G}_i$.

- Let $a \in A$ be an arrow. Suppose we are given bases $v^i_1, ..., v^i_{p_i}$ of V_i for $i = t(a)$ and $i = h(a)$. Then, one has the following identities

  $$\begin{aligned} &\frac{p_{t(a)} - \sigma_{t(a)}(m)}{r_{t(a)}\sigma_{t(a)}(m)} \frac{1}{p_{t(a)}} \Big(p_{t(a)} \mathrm{rk}\, \mathscr{E}_{t(a)}^{(j_{t(a)})} - j_{t(a)} r_{t(a)} \Big) - \frac{j_{t(a)}}{p_{t(a)}} \\ &\qquad = \frac{p_{t(a)} \mathrm{rk}\, \mathscr{E}_{t(a)}^{(j_{t(a)})}}{r_{t(a)}\sigma_{t(a)}(m)} - \frac{\mathrm{rk}\, \mathscr{E}_{t(a)}^{(j_{t(a)})}}{r_{t(a)}} - \frac{j_{t(a)}}{\sigma_{t(a)}(m)} \end{aligned}$$

  and (7)

  $$\begin{aligned} &\frac{p_{h(a)} + \sigma_{h(a)}(m)}{r_{h(a)}\sigma_{h(a)}(m)} \frac{1}{p_{h(a)}} \Big(p_{h(a)} \mathrm{rk}\, \mathscr{E}_{h(a)}^{(j_{h(a)})} - j_{h(a)} r_{h(a)} \Big) + \frac{j_{h(a)}}{p_{h(a)}} \\ &\qquad = \frac{p_{h(a)} \mathrm{rk}\, \mathscr{E}_{h(a)}^{(j_{h(a)})}}{r_{h(a)}\sigma_{h(a)}(m)} + \frac{\mathrm{rk}\, \mathscr{E}_{h(a)}^{(j_{h(a)})}}{r_{h(a)}} - \frac{j_{h(a)}}{\sigma_{h(a)}(m)}. \end{aligned}$$

Let $\underline{t} := \big([k_i : V_i \otimes \mathscr{O}_X(-m) \longrightarrow \mathscr{E}_i], i \in V; [\varphi_a], a \in A\big)$ and $\underline{g} := \mathrm{Gies}(\underline{t}) = \big([w_i], i \in V; [f_a], a \in A\big) \in \mathfrak{G}$.

By Theorem 3.3 and Formula (6), the point $\underline{g}$ is (semi)stable w.r.t. the linearization in $\mathscr{O}(l_1, ..., l_n; b_a, a \in A)$, if and only if, for every possible choice of bases $v^i_1, ..., v^i_{p_i}$ for V_i and indices $j_i \in \{0, ..., p_i\}$ with

$$f_a\big(V_{t(a)}^{(j_{t(a)})}\big) \quad \subset \quad V_{h(a)}^{(j_{h(a)})} \quad \text{for every arrow } a \in A,$$

one has

$$\begin{aligned} 0(\geq) \textstyle\sum_a b_a \Bigg[&\frac{p_{t(a)} - \sigma_{t(a)}(m)}{r_{t(a)}\sigma_{t(a)}(m)} \frac{1}{p_{t(a)}} \big(p_{t(a)} \mathrm{rk}\, \mathscr{E}_{t(a)}^{(j_{t(a)})} - j_{t(a)} r_{t(a)} \big) - \frac{j_{t(a)}}{p_{t(a)}} \\ &+ \frac{p_{h(a)} + \sigma_{h(a)}(m)}{r_{h(a)}\sigma_{h(a)}(m)} \frac{1}{p_{h(a)}} \big(p_{h(a)} \mathrm{rk}\, \mathscr{E}_{h(a)}^{(j_{h(a)})} - j_{h(a)} r_{h(a)} \big) + \frac{j_{h(a)}}{p_{h(a)}} \Bigg]. \end{aligned} \tag{8}$$

Applying Formula (7) and multiplying the result by $\sigma(m) = \sigma_1(m) \cdot \ldots \cdot \sigma_n(m)$, (8) becomes equivalent to

$$0 \quad (\geq) \quad \sum_{a\in A} b_a \Bigg[\check{\sigma}_{t(a)}(m) \Big\{ j_{t(a)} - \operatorname{rk} \mathscr{E}^{(j_{t(a)})}_{t(a)} \frac{\chi(\mathscr{E}_{t(a)}(m)) - \sigma_{t(a)}(m)}{\operatorname{rk} \mathscr{E}_{t(a)}} \Big\} \\ + \check{\sigma}_{h(a)}(m) \Big\{ j_{h(a)} - \operatorname{rk} \mathscr{E}^{(j_{h(a)})}_{h(a)} \frac{\chi(\mathscr{E}_{h(a)}(m)) + \sigma_{h(a)}(m)}{\operatorname{rk} \mathscr{E}_{h(a)}} \Big\} \Bigg]. \tag{9}$$

Assume first that $\underline{\mathrm{R}} := (\mathscr{E}_i, i \in V; \varphi_a, a \in A)$ is ϑ-(semi)stable. It follows from Theorem 4.4 that $\underline{\mathrm{R}}(m)$ fulfills the condition of $(\sigma_i(m), i \in V; b_a, a \in A)$-sectional (semi)stability for all sub-representations $(\mathscr{F}_i(m), i \in V; \varphi'_a(m), a \in A)$ for which $\mathscr{F}_i(m)$ is globally generated for all $i \in V$, in particular, for the sub-representation $(\mathscr{E}^{(j_i)}_i(m), i \in V; \varphi_{a|\mathscr{E}^{(j_{t(a)})}_{t(a)}}(m), a \in A)$. Since $j_i \leq h^0(\mathscr{E}^{(j_i)}_i(m))$ for all $i \in V$, this implies (9).

Now, suppose $\underline{\mathrm{g}}$ is (semi)stable for the given linearization and let $(\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$ be a sub-representation of $\underline{\mathrm{R}}$ for which $\mathscr{F}_i(m)$ is globally generated for all $i \in V$. Choose bases $v^i_1, \ldots, v^i_{p_i}$ of the V_i for which there are indices j_i with $H^0(k_i(m))(V^{(j_i)}_i) = H^0(\mathscr{F}_i(m))$, $i \in V$. Obviously,

$$f_a\big(V^{(j_{t(a)})}_{t(a)}\big) \quad \subset \quad V^{(j_{h(a)})}_{h(a)} \quad \text{for every arrow } a \in A.$$

Thus, (8) shows that $(\sigma_i(m), i \in V; b_a, a \in A)$-sectional (semi)stability is verified for $(\mathscr{F}_i(m), i \in V; \varphi'_a(m), a \in A)$. By Theorem 4.4, this implies that $\underline{\mathrm{R}}$ is ϑ-(semi)stable.

To see the assertion about the polystable points, we first remark that a point $\mathfrak{g} := \operatorname{Gies}([k_i], i \in V; [\varphi_a], a \in A) \in \mathfrak{G}^{ss}$ fails to be stable if and only if there is a destabilizing sub-representation $(\mathscr{F}_i, i \in V; \varphi'_a, a \in A)$. As explained before, this gives rise to a certain one parameter subgroup λ. Take $\lim_{z\to\infty} \lambda(z)([k_i], i \in V; [\varphi_a], a \in A)$. Then, it is not hard to see that the representation corresponding to that point is $(\mathscr{F}_i, i \in V; \varphi'_a, a \in A) \oplus (\mathscr{E}_i/\mathscr{F}_i, i \in V; \overline{\varphi}_a, a \in A)$. From the already proven semistable version of our theorem, it follows that this representation is again semistable and that $\lim_{z\to\infty} \lambda(z)\mathfrak{g}$ is a semistable point. Hence, it is clear that $\mathfrak{g}$ will be a polystable point if and only if every destabilizing sub-representation of $\underline{\mathrm{R}}$ is a direct summand, or, in other words, $\underline{\mathrm{R}}$ is ϑ-polystable.

ii) We will follow [7], Prop. 4.4.2, and use the valuative criterion of properness. Let $(C, 0) := \operatorname{Spec} R$ where R is a discrete valuation ring. By assumption, we are given a map $h: C \longrightarrow \mathfrak{G}^{ss}$ which lifts over $C \setminus \{0\}$ to $\mathfrak{T}^{\vartheta-ss}_{\underline{\mathrm{P}}}$. In particular, by the universal property of $\mathfrak{T}^{\vartheta-ss}_{\underline{\mathrm{P}}}$, there is a family

$$\big(V_i \otimes \pi_X^* \mathscr{O}_X(-m) \longrightarrow \mathfrak{E}^0_i, i \in V; \varphi^0_a, a \in A\big)$$

parametrized by $C \setminus \{0\}$. This can be extended to a certain family

$$\big(V_i \otimes \pi_X^* \mathscr{O}_X(-m) \longrightarrow \widetilde{\mathfrak{E}}_i, i \in V; \widetilde{\varphi}_a, a \in A\big)$$

on $C \times X$. Here, $V_i \otimes \pi_X^* \mathscr{O}_X(-m) \longrightarrow \widetilde{\mathfrak{E}}_i$ are families of not necessarily torsion free quotients with Hilbert polynomial P_i, $i \in V$, and $\widetilde{\varphi}_a \in \operatorname{Hom}\big(\widetilde{\mathfrak{E}}_{t(a)}, \widetilde{\mathfrak{E}}_{h(a)}\big)$. For each arrow $a \in A$, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\mathfrak{E}}_{t(a)} & \longrightarrow & \widetilde{\mathfrak{E}}^{\vee\vee}_{t(a)} & \longrightarrow & \mathfrak{N}_{t(a)} & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \widetilde{\varphi}_a} & & \downarrow{\scriptstyle \widetilde{\varphi}_a^{\vee\vee}} & & \downarrow{\scriptstyle \overline{\varphi}_a} & & \| \\
0 & \longrightarrow & \widetilde{\mathfrak{E}}_{h(a)} & \longrightarrow & \widetilde{\mathfrak{E}}^{\vee\vee}_{h(a)} & \longrightarrow & \mathfrak{N}_{h(a)} & \longrightarrow & 0.
\end{array}$$

As in [7], define $\mathfrak{N}'_i \subset \mathfrak{N}_i$, $i \in V$, as the union of the kernels of the multiplications by t^n, $n \in \mathbb{N}$, t a generator of the maximal ideal of R. Next, we set $\mathfrak{E}_i := \ker\big(\widetilde{\mathfrak{E}}_i^{\vee\vee} \longrightarrow \mathfrak{N}_i/\mathfrak{N}'_i\big)$, $i \in V$. These are C-flat families of torsion free coherent sheaves on $C \times X$. Since $\overline{\varphi}_a$ maps $\mathfrak{N}'_{t(a)}$ to $\mathfrak{N}'_{h(a)}$, the map $\widetilde{\varphi}_a^{\vee\vee}$ induces a homomorphism $\varphi_{C,a}\colon \mathfrak{E}_{t(a)} \longrightarrow \mathfrak{E}_{h(a)}$. By construction, there are homomorphisms $\mathfrak{k}_i\colon V_i \otimes \pi_X^*\mathscr{O}_X(-m) \longrightarrow \mathfrak{E}_i$, $i \in V$, which coincide on $(C \setminus \{0\}) \times X$ with the quotients we started with and which become generically surjective when restricted to $\{0\} \times X$. The family

$$\big(\mathfrak{k}_i\colon V \otimes \pi_X^*\mathscr{O}_X(-m) \longrightarrow \mathfrak{E}_i, i \in V; \varphi_{C,a}, a \in A\big)$$

defines a morphism of C to $\mathfrak{G}$ which is, of course, the map h of the beginning.

Let $k_i\colon V_i \otimes \mathscr{O}_X(-m) \longrightarrow \mathscr{E}_i$ be the restriction of $\mathfrak{k}_i$ to $\{0\} \times X$, $i \in V$, and $[f_a\colon V_{t(a)} \longrightarrow V_{h(a)}, a \in A]$ the $\mathfrak{P}$-component of $h(0)$. We claim that $H^0(k_i(m))$ must be injective for all $i \in V$. To see this, set $K_i := \ker(V_i \longrightarrow H^0(\mathscr{E}_i(m)))$, $i \in V$, and assume that not all the K_i are trivial. For each i, let $v^i_1, \ldots, v^i_{j_i}$ be a basis for K_i and complete it to a basis $v^i_1, \ldots, v^i_{p_i}$ of V_i. It follows from the construction that $f_a(K_{t(a)}) \subset K_{h(a)}$ for all arrows $a \in A$. Therefore, evaluating the semistability condition yields

$$0 \quad \geq \quad \sum_{a \in A} b_a\big(\check{\sigma}_{t(a)}(m) j_{t(a)} + \quad \check{\sigma}_{h(a)}(m) j_{h(a)}\big)$$

which is impossible.

Using now $H_i := k_i(V_i) \subset H^0(\mathscr{E}_i(m))$, one can check with the same methods as before that $(\mathscr{E}_i(m), i \in V; \varphi_{C,a|\{0\}\times X}(m), a \in A)$ is sectional semistable w.r.t. the parameters $(\sigma_i(m), i \in V; b_a, a \in A)$. But this implies $h^0(\mathscr{E}_i(m)) = p_i$ for $i \in V$, so that all the $\mathfrak{k}_i$ are honest quotients. This means that

$$\big(\mathfrak{k}_i\colon V \otimes \pi_X^*\mathscr{O}_X(-m) \longrightarrow \mathfrak{E}_i, i \in V; \varphi_{C,a}, a \in A\big)$$

defines a morphism $C \longrightarrow \mathfrak{T}$ which maps by our previous calculations to $\mathfrak{T}^{\vartheta-ss}_{\underline{\mathrm{P}}}$, thus providing the desired lifting of h.

References

[1] L. Álvarez-Cónsul, O. García-Prada, *Dimensional reduction,* $\mathrm{SL}(2,\mathbb{C})$*-equivariant bundles and stable holomorphic chains*, Int. J. Math. (to appear).

[2] D. Banfield, *The Geometry of Coupled Equations in Gauge Theory*, PhD thesis, Oxford, 1996.

[3] St. Bradlow, O. Garcia-Prada, *Stable triples, equivariant bundles and dimensional reduction*, Math. Ann. **304** (1996), 225-52.

[4] P. Gabriel, A.V. Roiter, *Representations of Finite-Dimensional Algebras*, Springer, 1992.

[5] O. Garcia-Prada, *Dimensional reduction of stable bundles, vortices and stable pairs*, Int. J. Math. **5** (1994), 1-52.

[6] N. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), 59-126.

[7] D. Huybrechts, M. Lehn, *The Geometry of the Moduli Spaces of Sheaves,* Vieweg, 1997.

[8] A. King, *Moduli of representations of finite dimensional algebras*, Quarterly J. Math. **45** (1994), 515-30.

[9] L. Le Bruyn, *Non-commutative geometry @n*, `http://win-www.uia.ac.be/u/lebruyn/`.

[10] G. Lusztig, *On quiver varieties*, Adv. Math. **136** (1998), 141-82.

[11] D. Mumford, *Geometric Invariant Theory*, Springer, 1965.

[12] I. Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kaehler fibrations*, `math.DG/9901076`.

[13] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. **91** (1998), 515-60.

[14] H. Nakajima, *Quiver varieties and finite dimensional representations of quantum affine algebras*, `math.QA/9912158`.

[15] N. Nitsure, *Moduli space of semistable pairs on a curve*, Proc. London Math. Soc. (3) **62** (1991), 275-300.

[16] Ch. Okonek, A. Schmitt, A. Teleman, *Master spaces for stable pairs*, Topology **38** (1999), 117-39.

[17] C.-M. Ringel, *Tame Algebras and Integral Quadratic forms*, Springer, 1984.

[18] A. Schmitt, *Projective moduli for Hitchin pairs*, Int. J. Math. **9** (1998), 107-18; Erratum **11** (1999), 589.

[19] A. Schmitt, *Framed Hitchin pairs*, Rev. roumaine math. pures appl. (to appear).

[20] A. Schmitt, *A universal construction for moduli spaces of decorated vector bundles*, `math.AG/0006029`.

[21] C. Simpson, *Moduli of representations of the fundamental group of a smooth manifold* I, Publ. Math. I.H.E.S. **79** (1994), 47-129.

[22] M.-S. Stupariu, *The Kobayashi-Hitchin Correspondence for Vortex-Type Equations Coupled with Higgs Fields*, PhD thesis, Zürich, 1998.

[23] K. Yokogawa, *Moduli of stable pairs*, J. Math. Kyoto Univ. **31** (1991), 311-27.

Universität GH Essen
FB 6 Mathematik und Informatik
D-45117 Essen
Deutschland

E-Mail: `alexander.schmitt@uni-essen.de`